MATH562 – Fall 2025

Problem Set: Week 12

1. A random sample $y_1, \ldots, y_n \overset{i.i.d.}{\sim} \mathbb{N}(\mu, \sigma^2)$ with average $\bar{y}$ is to be used to test the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $\mu = \mu_1$; below σ^2 is known and $z_p = \Phi^{-1}(p)$.

   (a) Show that if $\mu_1 > \mu_0$ then the most powerful critical region of size α is

   $$\mathcal{Y}_\alpha^+ = \left\{ y \in \mathbb{R}^n : \bar{y} \geq \mu_0 + \sigma n^{-1/2} z_{1-\alpha} \right\},$$

   and find the corresponding most powerful critical region $\mathcal{Y}_\alpha^-$ when $\mu_1 < \mu_0$.

   Solution: The most powerful test against any fixed value of $\mu \neq \mu_0$ is obtained from the Neyman–Pearson lemma. The likelihood ratio for testing $\mu = \mu_0$ against $\mu = \mu_1$ with σ known is

   $$\begin{aligned} \frac{f_1(y_1, \ldots, y_n)}{f_0(y_1, \ldots, y_n)} &= \frac{(2\pi\sigma^2)^{-n/2} \exp\{-\frac{1}{2\sigma^2} \sum_{j=1}^n (y_j - \mu_1)^2\}}{(2\pi\sigma^2)^{-n/2} \exp\{-\frac{1}{2\sigma^2} \sum_{j=1}^n (y_j - \mu_0)^2\}} \\ &= \exp\left[\frac{1}{2\sigma^2} \left\{2n\bar{y}(\mu_1 - \mu_0) - \mu_1^2 + \mu_0^2\right\}\right]. \end{aligned}$$

   This is monotone increasing in $\bar{y}$ for any fixed $\mu_1 > \mu_0$, and so the critical region rejects H_0 when $\bar{y} \geq t_\alpha$, with t_α chosen to give a test of size α. The null distribution of $\bar{Y}$ is $\mathbb{N}(\mu_0, \sigma^2/n)$, so

   $$\alpha = \text{Pr}_0(\bar{Y} \geq t_\alpha) = \text{Pr}_0\left\{n^{1/2}(\bar{Y} - \mu_0)/\sigma \geq n^{1/2}(t_\alpha - \mu_0)/\sigma\right\} = 1 - \Phi\left\{n^{1/2}(t_\alpha - \mu_0)/\sigma\right\},$$

   which implies that $n^{1/2}(t_\alpha - \mu_0)/\sigma = z_{1-\alpha}$, giving $t_\alpha = \mu_0 + \sigma n^{-1/2} z_{1-\alpha}$ and thus $\mathcal{Y}_\alpha^+$, as required. When $\mu_1 < \mu_0$, a similar computation leads to

   $$\mathcal{Y}_\alpha^- = \left\{(y_1, \ldots, y_n) : \bar{y} \leq \mu_0 + \sigma n^{-1/2} z_\alpha\right\}.$$

   (b) Are $\mathcal{Y}_\alpha^+$ and $\mathcal{Y}_\alpha^-$ uniformly most powerful against their respective alternatives? Explain. **Solution**: The critical region $\mathcal{Y}_\alpha^+$ is most powerful for any $\mu_1 > \mu_0$, so it is uniformly most powerful for $\mu_1 > \mu_0$, and likewise for $\mathcal{Y}_\alpha^-$ against the alternatives $\mu < \mu_0$.

   (c) Now consider the two-sided alternative $H : \mu_1 \neq \mu_0$. Compute the size of the critical region

   $$\mathcal{Y}_\beta = \left\{y \in \mathbb{R}^n : n^{1/2}|\bar{y} - \mu_0|/\sigma \geq z_{1-\beta}\right\}$$

   and hence give a two-sided critical region of size α. Is this uniformly most powerful against H? **Solution**: Symmetry of the distribution of $\bar{Y} - \mu_0$ under the null hypothesis implies that $\mathcal{Y}_\beta$ has size

   $$\text{Pr}_0(Y \in \mathcal{Y}_\beta) = \text{Pr}_0\left(n^{1/2}|\bar{Y} - \mu_0|/\sigma \geq z_{1-\beta}\right) = 2\,\text{Pr}_0\left\{n^{1/2}(\bar{Y} - \mu_0)/\sigma \geq z_{1-\beta}\right\} = 2\beta,$$

   so we should choose $\beta = \alpha/2$ to achieve size α. $\mathcal{Y}_{\alpha/2}$ is not uniformly most powerful of size α, because if $\mu_1 > \mu_0$ then $\mathcal{Y}_\alpha^+$ also has size α but has higher power (because $z_{1-\alpha} \leq z_{1-\alpha/2}$).

2. Consider the order statistics $0 < Y_{(1)} < \cdots < Y_{(n)}$ of a random sample $Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} \exp(\lambda)$.

(a) Show that $\min(Y_1,\ldots,Y_r)\sim\exp(r\lambda)$, and that each Y_j has the lack-of-memory property

$$\Pr(Y-x>y\mid Y>x)=\Pr(Y>y),\quad x,y>0.$$

Solution: As $\min(Y_1,\ldots,Y_r)>x$ if and only if $Y_1>x,\ldots,Y_r>x$, we have

$$\Pr\{\min(Y_1,\ldots,Y_r)\le x\}=1-\Pr\{\min(Y_1,\ldots,Y_r)>x\}=1-\Pr(Y_1>x)^r=1-\exp(-r\lambda x),\quad x>0,$$

and for $x,y>0$, $\Pr(Y-x>y\mid Y>x)$ equals

$$\frac{\Pr(Y-x>y,Y>x)}{\Pr(Y>x)}=\frac{\Pr(Y>y+x)}{\Pr(Y>x)}=\exp\{-\lambda(x+y)\}/\exp(-\lambda x)=\exp(-\lambda y),$$

as required.

(b) Show that $Y_j\stackrel{d}{=}E_j/\lambda$ with $E_1,\ldots,E_n i.i.d.\exp(1)$, and hence obtain the *Renyi representation*

$$Y_{(r)}\stackrel{d}{=}\frac{1}{\lambda}\sum_{j=1}^{r}\frac{E_j}{n+1-j},\quad r=1,\ldots,n.$$

Solution: As $\Pr(E_j/\lambda\le x)=\Pr(E_j\le\lambda x)=1-\exp(-\lambda x)=\Pr(Y_j\le x)$, we have $Y_j\stackrel{d}{=}E_j/\lambda$. We argue as follows: $Y_{(1)}$ is the smallest of n independent exponential variables, so it is exponential with parameter $n\lambda$ and therefore we can write $Y_{(1)}\stackrel{d}{=}E_1/(n\lambda)$;
the remaining $n-1$ variables have the lack of memory property, so given that $Y_{(1)}=x$ the remaining Y_j-x have exponential distributions with parameter λ. Thus $Y_{(2)}-Y_{(1)}$ is the minimum of $n-1$ exponential variables, i.e., $Y_{(2)}-Y_{(1)}\stackrel{d}{=}E_2/\{(n-1)\lambda\}$;
iterating the argument by successively conditioning on $Y_{(2)},\ldots,Y_{(n-1)}$ and obtaining the distributions of $Y_{(3)}-Y_{(2)},\ldots,Y_{(n)}-Y_{(n-1)}$ gives the stated representation.

(c) Find the means and covariances of $Y_{(1)},\ldots,Y_{(n)}$. **Solution**: A standard exponential variable has mean and variance both equal to 1, so

$$\mathrm{E}(Y_{(r)})=\frac{1}{\lambda}\sum_{j=1}^{r}\frac{1}{n+1-j},\quad \mathrm{Cov}(Y_{(r)},Y_{(s)})=\frac{1}{\lambda^2}\sum_{j=1}^{m}\frac{1}{(n+1-j)^2},\quad r,s,\in\{1,\ldots,n\},$$

with $m=\min(s,r)$ and the second formula giving the variance when $r=s$. Note the simple approximate integral formulae

$$\begin{aligned}\mathrm{E}(Y_{(r)}) &\doteq \frac{1}{\lambda}\int_{n-r+\frac12}^{n+\frac12}\frac{\mathrm{d}x}{x}=\lambda^{-1}\log\{(n+\frac12)/(n-r+\frac12)\},\\ \mathrm{Cov}(Y_{(r)},Y_{(s)}) &\doteq \frac{1}{\lambda^2}\int_{n-m+\frac12}^{n+\frac12}\frac{\mathrm{d}x}{x^2}=\lambda^{-2}\frac{m}{(n+\frac12)(n-m+\frac12)}.\end{aligned}$$

3. Below we consider different ways to combine evidence from independent p-values $P_1,\ldots,P_n$ from testing the same null hypothesis.

(a) Find the distributions of $-\log P_j$ and hence of $S_F=-\sum_j\log P_j$ (Fisher's statistic) and $S_P=-\sum_j\log(1-P_j)$ (Pearson's statistic). Give the size α critical regions for tests based on S_F and S_P. **Solution**: Clearly if P is small then $-\log P$ is large, and

$$\Pr(-\log P\le x)=\Pr(P\ge e^{-x})=1-e^{-x},\quad x>0,$$

so $-\log P$ has a standard exponential distribution. Thus S_F, a sum of independent exponential variables, has a gamma distribution, with upper tail probability

$$\Pr_0(S_F \le s) = \int_0^s \frac{x^{n-1}}{n!} e^{-x}\, \mathrm{d}x,$$

and quantiles s_α, say. The critical region is $\mathcal{Y}_1^F = \{(p_1, \ldots, p_n) \in (0,1)^n : -\sum_{j=1}^n \log p_j \ge s_{1-\alpha}\}$.

As $-\log P \stackrel{d}{=} -\log(1-P)$, $S_P \stackrel{d}{=} S_F$, giving critical region $\mathcal{Y}_1^P = \{(p_1, \ldots, p_n) \in (0,1)^n : -\sum_{j=1}^n \log(1 - p_j) \ge s_{1-\alpha}\}$.

(b) Give the size α critical region for a test based on $S_T = \min_j P_j$ (Tippett's statistic). **Solution**: Here $\Pr_0(S_T > s) = \Pr_0(P_1 > s, \ldots, P_n > s) = (1-s)^n$ for $s \in (0,1)$, and the critical region is given by $\Pr_0(S_T \le s_\alpha) = 1 - \Pr_0(S_T > s) = \alpha$. Therefore, we obtain $s_\alpha = 1 - (1-\alpha)^{1/n}$, giving $\mathcal{Y}_1^T = \{(p_1, \ldots, p_n) \in (0,1)^n : \min_j p_j \le 1 - (1-\alpha)^{1/n}\}$.

(c) Suppose that the alternative is such that $\Pr(P \le x) = x^{1/\gamma}$ for $x \in (0,1)$ and some $\gamma > 1$. Which of S_F, S_P and S_T is preferable, and why? **Solution**: Under this alternative we have

$$\Pr(-\log P \le x) = \Pr(P \ge e^{-x}) = 1 - (e^{-x})^{1/\gamma} = 1 - e^{-x/\gamma},$$

so $-\log P \sim \exp(1/\gamma)$ with $\gamma > 1$. This is an exponential family and we are comparing the simple null and alternative hypotheses $\gamma = 1$ and $\gamma > 1$, so Example 30 of the notes applies. The likelihood ratio for $p_1, \ldots, p_n$ is

$$\frac{f_1(p)}{f_0(p)} = \frac{\gamma^{-n} \prod_{j=1}^n p_j^{1/\gamma - 1}}{\prod_{j=1}^n 1} = \exp\left\{ -\sum_{j=1}^n \log p_j (1 - 1/\gamma) - n \log \gamma \right\},$$

which is an exponential family with $\varphi = -1/\gamma < 0$, $s^* = -\sum_j \log p_j$, $k(\varphi) = n \log \gamma = -n \log(-\varphi)$ and $\log m^*(p) = -\sum_j \log p_j$. Since φ is a monotone increasing function of γ, the computation in the example implies that the most powerful test has a critical region of the form $s^* > s_{1-\alpha}$, and therefore S_F is the best test statistic in this situation. As we always have $\gamma > 1$ or equivalently $\varphi < -1$ under the alternative, it is also uniformly most powerful.

(d) If P has density $x^{a-1}(1-x)^{b-1}/B(a,b)$, where $0 < x < 1$, $0 < a < 1$, $b \ge 1$ and $a \ne b$, show that the uniformly most powerful test involves $wS_F + (1-w)S_P$, where $w = (1-a)/(b-a)$. **Solution**: This extends (c), with the log likelihood ratio turning out to be

$$(a-1) \sum \log p_j + (b-1) \sum \log(1 - p_j) = (b-a) \{ wS_F + (1-w) S_P \}.$$

(e) What would you do if it is believed that the null hypothesis holds in a proportion $1-q$ of the tests and the alternative in (c) holds in the remaining ones, with both q and γ unknown? **Solution**: In this situation the cumulative distribution function for P is $(1-q)x + qx^{1/\gamma}$, so the density is $(1-q) + (q/\gamma)x^{1/\gamma - 1}$, for $x \in (0,1)$. As $\gamma > 1$, this implies that that the density is unbounded as $x \to 0$, which may not be so plausible, but in any case the obvious approach would be to estimate q and γ (for example using maximum likelihood) and hence decide whether $q = 0$ or $\gamma > 1$. In this case S_T seems attractive, because it seems likely that it would be able to profit from the spike under the alternative.