MATH562 – Fall 2025

Problem Set: Week 12

1. A random sample $y_1, \ldots, y_n \overset{i.i.d.}{\sim} \mathbb{N}(\mu, \sigma^2)$ with average $\bar{y}$ is to be used to test the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $\mu = \mu_1$; below σ^2 is known and $z_p = \Phi^{-1}(p)$.

   (a) Show that if $\mu_1 > \mu_0$ then the most powerful critical region of size α is
   $$\mathcal{Y}_\alpha^+ = \left\{ y \in \mathbb{R}^n : \bar{y} \geq \mu_0 + \sigma n^{-1/2} z_{1-\alpha} \right\},$$
   and find the corresponding most powerful critical region $\mathcal{Y}_\alpha^-$ when $\mu_1 < \mu_0$.

   (b) Are $\mathcal{Y}_\alpha^+$ and $\mathcal{Y}_\alpha^-$ uniformly most powerful against their respective alternatives? Explain.

   (c) Now consider the two-sided alternative $H : \mu_1 \neq \mu_0$. Compute the size of the critical region
   $$\mathcal{Y}_\beta = \left\{ y \in \mathbb{R}^n : n^{1/2} |\bar{y} - \mu_0| / \sigma \geq z_{1-\beta} \right\}$$
   and hence give a two-sided critical region of size α. Is this uniformly most powerful against H?

2. Consider the order statistics $0 < Y_{(1)} < \cdots < Y_{(n)}$ of a random sample $Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} \exp(\lambda)$.

   (a) Show that $\min(Y_1, \ldots, Y_r) \sim \exp(r\lambda)$, and that each Y_j has the lack-of-memory property
   $$\Pr(Y - x > y \mid Y > x) = \Pr(Y > y), \quad x, y > 0.$$

   (b) Show that $Y_j \overset{d}{=} E_j/\lambda$ with $E_1, \ldots, E_n i.i.d. \exp(1)$, and hence obtain the *Renyi representation*
   $$Y_{(r)} \overset{d}{=} \frac{1}{\lambda} \sum_{j=1}^{r} \frac{E_j}{n+1-j}, \quad r = 1, \ldots, n.$$

   (c) Find the means and covariances of $Y_{(1)}, \ldots, Y_{(n)}$.

3. Below we consider different ways to combine evidence from independent p-values $P_1, \ldots, P_n$ from testing the same null hypothesis.

   (a) Find the distributions of $-\log P_j$ and hence of $S_F = -\sum_j \log P_j$ (Fisher's statistic) and $S_P = -\sum_j \log(1-P_j)$ (Pearson's statistic). Give the size α critical regions for tests based on S_F and S_P.

   (b) Give the size α critical region for a test based on $S_T = \min_j P_j$ (Tippett's statistic).

   (c) Suppose that the alternative is such that $\Pr(P \leq x) = x^{1/\gamma}$ for $x \in (0, 1)$ and some $\gamma > 1$. Which of S_F, S_P and S_T is preferable, and why?

   (d) If P has density $x^{a-1}(1-x)^{b-1}/B(a, b)$, where $0 < x < 1$, $0 < a < 1$, $b \geq 1$ and $a \neq b$, show that the uniformly most powerful test involves $wS_F + (1-w)S_P$, where $w = (1-a)/(b-a)$.

   (e) What would you do if it is believed that the null hypothesis holds in a proportion $1-q$ of the tests and the alternative in (c) holds in the remaining ones, with both q and γ unknown?