MATH562 – Fall 2025

Problem Set: Week 11

1. Suppose we hope to eliminate to nuisance parameters λ from a likelihood $L(\psi, \lambda)$ by using an integrated likelihood
$$\int L(\psi, \lambda)\ \mathrm{d}\lambda.$$
   (a) Criticise this approach by computing the integrated likelihoods when the likelihood is based on two independent exponential variables with parameters (i) λ and $\lambda\psi$, (ii) $1/\lambda$ and ψ/λ, where $\lambda, \psi > 0$.

   (b) Now suppose that in (i) the parameters are given a density $\pi(\psi, \lambda)$ and we compute the resulting marginal density for ψ. Show that if the corresponding prior density is used in the parametrization in (ii), the problems in (a) do not arise.

*2. A random sample $y_1, \ldots, y_n$ of distinct observations, $y_i \neq y_j$ for $i \neq j$, has arisen from an unknown distribution function G. Consider a multinomial distribution in which $p_j = \Pr(Y = y_j)$, for $j = 1, \ldots, n$.

   (a) Use Lagrange multipliers to show that the empirical distribution function with $p_j \equiv 1/n$ maximises the likelihood $\sum_j \log p_j$ of the observed data subject to the constraints $p_j \geq 0$ and $\sum_{j=1}^n p_j \leq 1$.
   Hint: first argue that its sufficient to consider the case where $p_j > 0$ for all j and $\sum_{j=1}^n p_j = 1$.

   (b) Now add the constraint that $\sum_j p_j c_j(\theta) = 0$, where $c_j(\theta) \equiv c(y_j; \theta)$ is a $d \times 1$ function of y_j and θ; this is the empirical version of the constraint $\mathrm{E}\{c(Y; \theta)\} = 0$, with expectation taken over $Y \sim G$. Show that in this case the log-likelihood for a specific θ, maximizing over the p_j given the constraints, is the *empirical log-likelihood*
$$\ell_{\mathrm{E}}(\theta) = -n \log n - \sum_{j=1}^{n} \log\{1 + \lambda_\theta^\top c_j(\theta)\}, \quad \text{where } \lambda_\theta \text{ satisfies} \quad 0 = \sum_{j=1}^{n} \frac{c_j(\theta)}{1 + \lambda_\theta^\top c_j(\theta)}.$$

   (c) If $c_j(\theta) = y_j - \theta$, show that the maximum empirical likelihood estimate is $\hat{\theta}_{\mathrm{E}} = \bar{y}$, with $\lambda_\theta = 0$, and that for any valid θ we have $\sum y_j p_j = \theta$ and $\min_j y_j < \theta < \max_j y_j$.

3. In testing $\alpha \neq 1$ when $Y_1, \ldots, Y_n$ is a random sample from the gamma (α, λ) distribution, with λ unknown (Problem 1(c) of Week 10),

   (a) show that the distribution of the test statistic is invariant to λ

   (b) describe how you would simulate the distribution of the test statistic in (a) to obtain p-values without relying on the asymptotic distribution for $n \to \infty$.

   (c) show that λ may be eliminated by appropriate conditioning.

4. (a) If $P \sim U(0,1)$, show that $Q = 2\min(P, 1-P) \sim U(0,1)$.

   (b) What are the achievable significance levels for testing that a single Poisson variable has mean $\mu_0 = 2$, with alternative mean (i) greater than 2 and (ii) less than 2? Does this matter for computing confidence intervals?