MATH562 – Fall 2025

Problem Set: Week 10

1. Let $Y_1, \ldots, Y_n$ be a random sample from the gamma density $\lambda^\alpha y^{\alpha-1}e^{-\lambda y}/\Gamma(\alpha)$, for $y > 0$ and $\alpha, \lambda > 0$.

   (a) Find the expected information matrix $\imath(\alpha, \lambda)$, in terms of the digamma function $\Psi(\alpha) = \mathrm{d}\log\Gamma(\alpha)/\mathrm{d}\alpha$ its derivative Ψ' (the trigamma function); note that $\Psi(1) \doteq -0.577$ and $\Psi'(1) \doteq 1.645$.. **Solution**: The log likelihood contribution from a single observation,

   $$\ell(\alpha, \lambda) = \alpha \log \lambda + (\alpha - 1)\log y - \lambda y - \log \Gamma(\alpha), \quad \alpha, \gamma > 0,$$

   has first derivatives $\ell_\alpha = \log\lambda + \log y - \Psi(\alpha)$, $\ell_\lambda = \alpha/\lambda - y$ and second derivatives $\ell_{\alpha\alpha} = -\Psi'(\alpha)$, $\ell_{\alpha\lambda} = 1/\lambda$, $\ell_{\lambda\lambda} = -\alpha/\lambda^2$, so the observed information matrix equals the expected information matrix,

   $$\imath(\alpha, \lambda) = n\imath_1(\alpha, \lambda) = n\begin{pmatrix} \Psi'(\alpha) & -1/\lambda \\ -1/\lambda & \alpha/\lambda^2 \end{pmatrix}.$$

   (b) Find the score statistic for testing whether $\alpha = 1$ when λ is known, and give its large-sample distribution. **Solution**: The score statistic is ℓ_α for the sample divided by the square root of its asymptotic variance $n\Psi'(\alpha)$, all evaluated at $\alpha = 1$; this gives

   $$\frac{\sum_{j=1}^n \log(\lambda Y_j) - n\Psi(1)}{\{n\Psi'(1)\}^{1/2}} \sim \mathbb{N}(0, 1).$$

   (c) Find the score statistic for testing whether $\alpha = 1$ when λ is unknown, and give its large-sample distribution. **Solution**: The score statistic is $\ell_\alpha^\mathsf{T} \jmath^{\alpha\alpha}\ell_\alpha$, evaluated at the maximum likelihood estimator when $\alpha = 1$, i.e., at $(\alpha, \hat\lambda_\alpha)|_{\alpha=1} = (1, 1/\bar{Y})$, and with

   $$\jmath^{\alpha\alpha} = (\hat\jmath_{\alpha\alpha} - \hat\jmath_{\alpha\lambda}\hat\jmath_{\lambda\lambda}^{-1}\hat\jmath_{\lambda\alpha})^{-1} = (\hat\imath_{\alpha\alpha} - \hat\imath_{\alpha\lambda}\hat\imath_{\lambda\lambda}^{-1}\hat\imath_{\lambda\alpha})^{-1} = \frac{1}{n\{\Psi'(\alpha) - 1/\alpha\}}$$

   evaluated at $\alpha = 1$. Hence the score statistic is

   $$\left\{\sum_{j=1}^n \log(Y_j/\bar{Y}) - n\Psi(1)\right\}^2 / \left[n\{\Psi'(1) - 1\}\right] \sim \chi_1^2;$$

   in fact here, since the interest parameter is scalar, we might use the approximation

   $$\frac{\sum_{j=1}^n \log(Y_j/\bar{Y}) - n\Psi(1)}{[n\{\Psi'(1) - 1\}]^{1/2}} \sim \mathbb{N}(0, 1).$$

   (d) Show that the parameter $\mu = \alpha/\lambda$ is orthogonal to α. Find the score statistic corresponding to (b) in this orthogonal parametrisation. Comment. **Solution**: The log likelihood function in terms of $\mu = \alpha/\lambda$ is obtained by setting $\lambda = \alpha/\mu$, and is

   $$\ell^*(\alpha, \mu) \equiv \alpha\log\alpha - \alpha\log\mu + (\alpha - 1)\log y - \alpha y/\mu - \log\Gamma(\alpha), \quad \alpha, \mu > 0,$$

   giving $\ell^*_{\alpha\mu} = y/\mu^2 - 1/\mu$, which has expectation zero; hence μ and α are orthogonal. This could also be found by using the Jacobian J for the transformation $(\alpha, \mu) \mapsto (\alpha, \lambda)$ to compute $\imath^*(\alpha, \mu) = J\imath(\alpha, \lambda)J$. In the orthogonal parametrisation $\ell^*_\alpha = 1 + \log(y/\mu) - y/\mu + \log\alpha - \Psi(\alpha)$ and $\ell^*_{\alpha\alpha} = 1/\alpha - \Psi'(\alpha)$. Now $\hat\mu_\alpha = \bar{Y}$, and owing to the orthogonality $\jmath^{\alpha\alpha} = \hat\jmath_{\alpha\alpha}^{-1} = 1/[n\{\Psi'(\alpha) - 1/\alpha\}]$, which gives $1/[n\{\Psi'(1) - 1\}]$ when $\alpha = 1$. The numerator of the score statistic is $\sum_{j=1}^n\{1 + \log(Y_j/\mu) - Y_j/\mu + \log\alpha - \Psi(\alpha)\}$ evaluated at $\alpha = 1$ and $\mu = \hat\mu_1 = \bar{Y}$, and the upshot is that the statistic is the same as in (b).

2. If Y_1 and Y_2 are independent exponential variables with means γ^{-1} and $(\gamma\psi)^{-1}$, show that a parameter $\lambda(\psi,\gamma)$ orthogonal to ψ solves the equation $\partial\gamma/\partial\psi = -\gamma/(2\psi)$, and (without solving this PDE, unless you feel the urge) verify that a possible solution is $\lambda = \gamma\psi^{1/2}$.

Solution:

The log likelihood in the non-orthogonal parametrization is

$$\ell^*(\psi,\gamma) = \log\gamma - \gamma y_1 + \log(\gamma\psi) - \gamma\psi y_2 = 2\log\gamma + \log\psi - \gamma(y_1 + \psi y_2), \quad \gamma,\psi > 0,$$

so its observed and Fisher information matrices have elements $-\ell^*_{\gamma\gamma} = 2/\gamma^2$, $-\ell^*_{\gamma\psi} = y_2$ and $-\ell^*_{\psi\psi} = 1/\psi^2$, and $\imath^*_{\gamma\gamma} = 2/\gamma^2$, $\imath^*_{\gamma\psi} = 1/(\gamma\psi)$ and $\imath^*_{\psi\psi} = 1/\psi^2$. Hence the partial differential equation giving the orthogonal parametrization is

$$\frac{\partial\gamma}{\partial\psi} = -\imath^{*-1}_{\gamma\gamma}(\psi,\gamma)\imath^*_{\gamma\psi}(\psi,\gamma) = -\frac{\gamma^2}{2}\frac{1}{\gamma\psi} = -\frac{\gamma}{2\psi}, \quad \gamma,\psi > 0,$$

as required.

To check that $\lambda = \gamma\psi^{1/2}$ is orthogonal to ψ, we write $\gamma = \lambda\psi^{-1/2}$ so the log likelihood becomes

$$\ell(\psi,\lambda) = 2\log\lambda - \lambda\psi^{-1/2}(y_1 + \psi y_2),$$

and note that $\ell_{\lambda\psi} = (y_1 - \psi y_2)/(2\psi^{3/2})$ has expectation $\{\gamma^{-1} - \psi/(\gamma\psi)\}/(2\psi^{3/2}) = 0$. Hence λ is orthogonal to ψ, as required.

The question only asks you to check that the given solution provides an orthogonal transformation, so the material below is included only to show how the PDE

$$2\psi\frac{\partial\gamma}{\partial\psi} = -\gamma$$

would be solved if the solution had not been given in the question. Now (e.g., Theorem 2, page 50 of Sneddon, *Elements of Partial Differential Equations*, 1957),

> "The general solution of the linear partial differential equation
>
> $$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R$$
>
> is $F(u,v) = 0$, where F is an arbitrary function and $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ form a solution of the equations
>
> $$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}."$$

In the present setting $z = \gamma$, $x = \psi$, $P = 2\psi$, $Q = 0$ and $R = -\gamma$, so we need to solve

$$\frac{\mathrm{d}\psi}{2\psi} + \frac{\mathrm{d}\gamma}{\gamma} = 0 \quad \Longrightarrow \quad \frac{1}{2}\log\psi + \log\gamma = c \quad \Longrightarrow \quad \gamma\psi^{1/2} = c,$$

and thus according to the theorem, the general solution is any function of $\gamma\psi^{1/2}$, such as $\lambda(\psi,\gamma) = \gamma\psi^{1/2}$.