INTRO TO DYNAMICAL SYSTEMS FALL 2025, PROBLEM SET 10

(1) Let $A \in \mathrm{Mat}\big(n \times n, \mathbb{R}\big)$ a matrix whose eigenvalues $\lambda \in \mathbb{C}$ satisfy
$$\alpha < \mathrm{Re}\,\lambda < \beta$$
for two real numbers α, β. Show that there is an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that
$$\alpha \big\|x\big\|^2 \leq \langle Ax, x \rangle \leq \beta \big\|x\big\|^2,$$
where $\big\|x\big\|^2 = \langle x, x \rangle$.

(2) Consider the system of ODEs
$$\dot{\underline{y}} = \underline{f}(\underline{y})$$
where as usual $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}$. Assume that $\underline{0} \in \mathbb{R}^n$ is a fixed point, i. e. $f(\underline{0}) = \underline{0}$, and $f \in C^1\big(\mathbb{R}^n, \mathbb{R}^n\big)$. Finally, assume that
$$D\underline{f}(\underline{0}),$$
the Jacobian matrix, has only eigenvalues with negative real part. By using (1), show directly (i. e. without using Hartman-Grobman) that there exist $C > 0, c > 0$ and $\delta > 0$ such that for any initial condition
$$\underline{y}(0),\ \big\|\underline{y}(0)\big\| < \delta,$$
we have that
$$\big\|\underline{y}(t)\big\| \leq C \cdot e^{-ct} \cdot \big\|\underline{y}(0)\big\|.$$

(3) Consider the system
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ x - x^3 \end{pmatrix}$$
Show that the origin is an unstable fixed point for its flow, and that the local stable and unstable manifold are actually part of one trajectory.