PROOF OF POINCARÉ-SIEGEL FOR THE SIEGEL CASE

We recall that our goal is to find a function h which is bi-holomorphic around $z = 0$ and such that

$$h^{-1} \circ f \circ h = \lambda \cdot z \tag{0.1}$$

for small enough $|z|$, where the holomorphic f is given by

$$f(z) = \lambda \cdot z + \sum_{k=2}^{\infty} a_k z^k,$$

and moreover $\lambda = e^{2\pi i \alpha}$ with $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ a diophantine number. Assume that we can approximately solve this equation so that

$$h^{-1} \circ f \circ h = \lambda \cdot z + u, \tag{0.2}$$

where $u = \sum_{k=2}^{\infty} b_k z^k$. Then we pick w so that

$$w\big(\lambda z\big) - \lambda \cdot w(z) = u. \tag{0.3}$$

Since u vanishes quadratically at $z = 0$, given $\varepsilon > 0$ we can pick $r > 0$ such that

$$\big|u'(z)\big| < \varepsilon,\ |z| < r. \tag{0.4}$$

Then we require precise pointwise control of w as well as w' as in the following

Lemma 0.1. *Assuming* (0.4)*, and letting* $0 < \rho \leq r$, $0 < \triangle < 1$, *we have*

$$\big|w(z)\big| < \varepsilon\rho \cdot C(c,d) \cdot \triangle^{-d-1},\ |z| \leq \rho \cdot (1 - \triangle),$$
$$\big|w'(z)\big| < \varepsilon \cdot \frac{C(c,d)}{1-\triangle} \cdot \triangle^{-d-1},\ |z| \leq r \cdot (1 - \triangle).$$

Proof. We rely on Lemma 4.2 from Lecture 9. Thanks to assumption (0.4), as well as $u(0) = 0$, we infer that

$$\big|u(z)\big| < \varepsilon \cdot \rho,\ |z| \leq \rho.$$

Then Lemma 4.2 implies that

$$\big|w(z)\big| < \varepsilon \cdot \rho \cdot C(c,d) \cdot \triangle^{-d-1}, |z| \leq \rho(1 - \triangle).$$

This is the first estimate asserted by the lemma.
For the second bound, observe that if (0.3) holds, then we also have

$$z\lambda w'\big(\lambda z\big) - \lambda \cdot zw'(z) = \big((\cdot) \cdot w'(\cdot)\big)(\lambda z) - \lambda \cdot zw'(z) = zu'$$

Again applying Lemma 4.2 we infer that

$$\big|z \cdot w'(z)\big| \leq \varepsilon \cdot \rho \cdot C(c,d) \cdot \triangle^{-d-1}, |z| \leq \rho(1 - \triangle).$$

In particular, choosing $|z| = \rho(1 - \triangle)$, we find

$$\big|w'(z)\big| \leq \varepsilon \cdot \frac{C(c,d)}{1-\triangle} \cdot \triangle^{-d-1},\ |z| = \rho(1 - \triangle).$$

The right hand bound is independent of ρ, so this holds for any $|z| \leq r(1 - \triangle)$. □

Keeping in mind that our goal is to replace (0.2) by an improved version

$$\big(\mathrm{id} + w\big)^{-1} \circ h^{-1} \circ f \circ h \circ \big(\mathrm{id} + w\big) = \lambda \cdot z + u_1, \tag{0.5}$$

we shall now have to establish two things

- Control the image of both $\big(\mathrm{id} + w\big)^{-1}$, $\big(\mathrm{id} + w\big)$ when restricted to small discs.

- Establish a much improved bound for u_1, and more precisely, that we essentially replace the smallness ε for u as in (0.4) by ε^2, at the expense of shrinking the disc a bit.

The following lemma takes care of the first point:

Lemma 0.2. *Assuming the smallness conditions*

$$C(c,d)\cdot\varepsilon < \triangle^{d+2},\ 0<\triangle<\frac{1}{4}$$

we have the properties (we use the shorthand $B_r := B_r(0)$)

$$\big(id+w\big)\big(B_{r(1-4\triangle)}\big)\subset B_{r(1-3\triangle)},\ B_{r(1-2\triangle)}\subset\big(id+w\big)\big(B_{r(1-\triangle)}\big).$$

Here r is such that (0.4) *holds.*

Proof. Taking advantage of Lemma 0.1, we have for $z\in B_{r(1-4\triangle)}$

$$\big|(\mathrm{id}+w)(z)\big|\le|z|+\big|w(z)\big|<r(1-4\triangle)+\varepsilon r\cdot C(c,d)\cdot\triangle^{-d-1}<r\cdot(1-3\triangle)$$

due to our assumption.
For the second part of the lemma, we recall Rouché's theorem from complex analysis, which asserts for our situation that if f,g are two holomorphic functions defined in the neighborhood of $\overline{B_{r(1-\triangle)}}$ and are such that

$$\big|g(z)\big|<\big|f(z)\big|$$

for all $z\in\partial B_{r(1-\triangle)}$, then the functions

$$f,f+g$$

have the same number of zeroes counted with multiplicity inside $B_{r(1-\triangle)}$. Now pick an arbitrary

$$z_0\in B_{r(1-2\triangle)}$$

and set

$$f=z-z_0,\ g=w(z).$$

Then we have

$$\big|f(z)\big|\ge r\cdot\triangle,\ \big|g(z)\big|<r\cdot\triangle$$

by our assumption for any $z\in\partial B_{r(1-\triangle)}$, and hence Rouché's theorem applies. This means there is exactly one $z\in B_{r(1-\triangle)}$ with

$$z+w(z)=z_0$$

for each $z_0\in B_{r(1-2\triangle)}$, proving the lemma. □

The preceding lemma allows us to make sense of

$$\big(\mathrm{id}+w\big)^{-1}\circ h^{-1}\circ f\circ h\circ\big(\mathrm{id}+w\big)=\big(\mathrm{id}+w\big)^{-1}\circ\big(\lambda\cdot\mathrm{id}+u\big)\circ\big(\mathrm{id}+w\big)$$

Lemma 0.3. *Assuming the bound* (0.4) *and further the smallness condition*

$$\varepsilon\cdot C(c,d)<\triangle^{d+2}\cdot(1-\triangle),\ 0<\varepsilon<\triangle<\frac{1}{5},$$

the map

$$\big(id+w\big)^{-1}\circ\big(\lambda\cdot id+u\big)\circ\big(id+w\big):B_{r(1-4\triangle)}\longrightarrow B_{r(1-\triangle)}$$

is defined and holomorphic.

Proof. From the preceding lemma we know that under the present assumptions, we have

$$\big(\mathrm{id}+w\big)\big(B_{r(1-4\triangle)}\big)\subset B_{r(1-3\triangle)}.$$

Next, we have

$$\Big|\big(\lambda\cdot\mathrm{id}+u\big)(z)\Big|\le r(1-3\triangle)+\varepsilon\cdot r\le r(1-2\triangle),\ z\in B_{r(1-3\triangle)}.$$

Again using the preceding lemma, the map

$$\big(\mathrm{id}+w\big)^{-1}:B_{r(1-2\triangle)}\longrightarrow B_{r(1-\triangle)}$$

is defined and holomorphic. Indeed, observe that $\mathrm{id} + w$ maps $B_{r(1-\triangle)}$ injectively onto its open image $U \supset B_{r(1-2\triangle)}$. To see this, observe that for $z, z' \in B_{r(1-\triangle)}$, we have

$$\begin{aligned}\big|w(z) - w(z')\big| &= \big|(z - z')\big| \cdot \Big| \int_0^1 w'(tz + (1-t)z')\,dt\Big| \\ &\leq \frac{1}{5} \cdot \big|(z - z')\big|.\end{aligned}$$

It follows that

$$\big|(\mathrm{id} + w)(z) - (\mathrm{id} + w)(z')\big| \geq \frac{4}{5} \cdot \big|(z - z')\big|,$$

which implies the injectivity. The fact that there is a holomorphic inverse

$$(\mathrm{id} + w)^{-1} : U \longrightarrow B_{r(1-\triangle)}$$

follows. □

We can then write

$$\big(\mathrm{id} + w\big)^{-1} \circ \big(\lambda \cdot \mathrm{id} + u\big) \circ \big(\mathrm{id} + w\big) = \lambda \cdot z + u_1,$$

and the linchpin is now to establish the following much improved estimate for u_1:

Lemma 0.4. *Under the same hypotheses as for the previous lemma, we have the estimate*

$$\big|u_1'(z)\big| \leq \varepsilon^2 \cdot \frac{5C(c,d)}{4\triangle^{d+2}}$$

provided $z \in B_{r(1-5\triangle)}$.

Note that we need to estimate the derivative u_1' since that was our starting point for u. Also, note that we have the much smaller parameter ε^2 here compared to ε before.

Proof. We write the equation defining u_1 in the form

$$\big(\mathrm{id} + w\big)\big(\lambda \cdot z + u_1\big) = \big(\lambda \cdot \mathrm{id} + u\big) \circ \big(\mathrm{id} + w\big),$$

which can be written more explicitly as

$$\lambda z + u_1(z) + w\big(\lambda \cdot z + u_1\big) = \lambda z + \lambda \cdot w(z) + u\big(z + w(z)\big)$$

We arrange this in the form of a fixed point equation for u_1:

$$\begin{aligned}u_1(z) &= -w\big(\lambda \cdot z + u_1\big) + \lambda \cdot w(z) + u\big(z + w(z)\big) \\ &= -w\big(\lambda z\big) + \lambda \cdot w(z) + u(z) + \big[w\big(\lambda z\big) - w\big(\lambda \cdot z + u_1\big)\big] \\ &\qquad + \big[u\big(z + w(z)\big) - u(z)\big]\end{aligned}$$

The sum of the first three terms on the right vanishes thanks to our choice of w. We next estimate the terms in parentheses at the end, for $z \in B_{r(1-4\triangle)}$. Note that by the preceding lemma, we have

$$\lambda z + u_1(z) \in B_{r(1-\triangle)}$$

for such z, and further any point on the straight line segment linking λz to $\lambda z + u_1(z)$ is in $B_{r(1-\triangle)}$. We conclude that

$$\begin{aligned}\Big|w\big(\lambda z\big) - w\big(\lambda \cdot z + u_1\big)\Big| &= \Big|\int_0^1 w'\big(\lambda \cdot z + t \cdot u_1\big) \cdot u_1(z)\,dt\Big| \\ &\leq \big|u_1(z)\big| \cdot \varepsilon \cdot \frac{C(c,d)}{1-\triangle} \cdot \triangle^{-d-1},\end{aligned}$$

where we have taken advantage of Lemma 0.1. Our assumptions then imply that the preceding can be bounded by

$$\big|u_1(z)\big| \cdot \varepsilon \cdot \frac{C(c,d)}{1-\triangle} \cdot \triangle^{-d-1} \leq \frac{\big|u_1(z)\big|}{5}.$$

We further have the estimate

$$\begin{aligned}\left|u\big(z+w(z)\big)-u(z)\right| &\leq \left|w(z)\right|\cdot\int_0^1\left|u'\big(z+tw(z)\big)\right|dt\\ &\leq \varepsilon r\cdot C(c,d)\cdot\triangle^{-d-1}\cdot\varepsilon.\end{aligned}$$

Here we have again used Lemma 0.1. We then find

$$\left|u_1(z)\right|\leq r\cdot\frac{5}{4}\cdot C(c,d)\cdot\triangle^{-d-1}\cdot\varepsilon^2,\ z\in B_{r(1-4\triangle)}.$$

Invoking the Cauchy formula, we can infer a derivative bound from this at the expense of shrinking the domain to $B_{r(1-5\triangle)}$:

$$\left|u_1'(z)\right|\leq\frac{5}{4}\cdot C(c,d)\cdot\triangle^{-d-2}\cdot\varepsilon^2,\ z\in B_{r(1-5\triangle)}.$$

□

Let us summarise what we have achieved thus far: starting from the assumption (0.4) and the relation (0.2), we have constructed a slightly modified

$$\widetilde{h}=h\circ\big(\mathrm{id}+w\big),$$

such that on the slightly smaller disc

$$B_{r(1-5\triangle)}$$

for some $0<\triangle<\frac{1}{5}$, we have the relation

$$\widetilde{h}^{-1}\circ f\circ\widetilde{h}=\lambda z+u_1,$$

where now u_1 satisfies the estimate stated in the preceding lemma. This estimate is much sharper than the one for u', provided ε is small enough in relation to $\triangle$.

But now we can re-start the whole process, with r replaced by $r(1-5\triangle)$ and u replaced by u_1. We then simply need to choose the parameters suitably that the whole process converges (very rapidly!) to the desired conjugation map. For this, we make the following choices:

$$r_n=\frac{r}{2}\cdot\big(1+2^{-n}\big)\tag{0.6}$$

where $r\leq 1$ is chosen as in (0.4) with $\varepsilon=\varepsilon_0$ sufficiently small, as determined below. Next, we set

$$\triangle_n=\frac{1}{10(2^n+1)},\ n\geq 1,\tag{0.7}$$

which implies that

$$\frac{r_{n+1}}{r_n}=\frac{1+2^{-n-1}}{1+2^{-n}}=1-\frac{2^{-n-1}}{1+2^{-n}}=1-5\triangle_n.$$

Now assuming that

$$\left|u_n'\right|<\varepsilon_n$$

on B_{r_n} with $n\geq 1$, then we get from the preceding lemma and the corresponding choice of w_n that

$$\left|u_{n+1}'(z)\right|\leq\varepsilon_n^2\cdot\frac{5C(c,d)}{4\triangle_n^{d+2}}\leq C_1(c,d)\cdot\big(2^n+1\big)^{d+2}\cdot\varepsilon_n^2,\ z\in B_{r_{n+1}}(0).$$

Thus we can set

$$\varepsilon_{n+1}=C_1(c,d)\cdot\big(2^n+1\big)^{d+2}\cdot\varepsilon_n^2.$$

It follows that if we introduce the new sequence

$$\gamma_n:=C_2(c,d)\cdot\big(2^n+1\big)^{d+2}\cdot\varepsilon_n,$$

where C_2 is chosen so that

$$C_2=M\cdot(C_1+1)\cdot 2^{d+2},$$

with $M \geq 1$ a large enough constant which we will chosen below, then we find that

$$\begin{aligned}\gamma_{n+1} &= C_1(c,d)\cdot\big(2^n+1\big)^{d+2}\cdot\big(2^{n+1}+1\big)^{d+2}\cdot C_2\cdot\varepsilon_n^2\\ &= C_2^{-1}\cdot C_1(c,d)\cdot\frac{\big(2^{n+1}+1\big)^{d+2}}{\big(2^n+1\big)^{d+2}}\cdot\gamma_n^2\\ &\leq \gamma_n^2.\end{aligned}$$

In particular if we pick the smallness constant ε_0 such that

$$C_2(c,d)\cdot\big(2^0+1\big)^{d+2}\cdot\varepsilon_0 < \frac{1}{2}, \tag{0.8}$$

then the γ_n and a fortiori the ε_n will converge faster than exponentially toward zero. Specifically, we obtain that

$$\big|\gamma_n\big| \leq 2^{-2^n},\ \big|\varepsilon_n\big| \leq C_2^{-1}\cdot 2^{-2^n}\big(2^n+1\big)^{-d-2} \leq M^{-1}\cdot 2^{-2^n}\big(2^n+1\big)^{-d-2}. \tag{0.9}$$

We observe that if M is chosen sufficiently large, then the smallness condition of Lemma 0.3 is automatically satisfied for $\triangle_n, \varepsilon$, $n \geq 0$. From Lemma 0.1 we deduce that

$$\big|w_n(z)\big| \leq C_3\cdot C_4^n\cdot 2^{-2^n}$$

for $z \in B_{r_{n+1}}(0)$ and suitable constants $C_{3,4}$.

The conjugating map h which achieves

$$h^{-1}\circ f\circ h(z) = \lambda\cdot z$$

on $B_{\frac{r}{2}}(0)$ is then given by

$$h = \lim_{n\to\infty}\big(\mathrm{id}+w_1\big)\circ\big(\mathrm{id}+w_2\big)\circ\ldots\circ\big(\mathrm{id}+w_n\big)$$

That this indeed converges on $B_{\frac{r}{2}}(0)$ follows by observing that (using Lemma 0.2)

$$\big(\mathrm{id}+w_1\big)\circ\big(\mathrm{id}+w_2\big)\circ\ldots\circ\big(\mathrm{id}+w_n\big)\Big|_{B_{r_{n+1}}(0)} : B_{r_{n+1}}(0)\longrightarrow B_r(0) \tag{0.10}$$

But then calling

$$h_n := \big(\mathrm{id}+w_1\big)\circ\big(\mathrm{id}+w_2\big)\circ\ldots\circ\big(\mathrm{id}+w_n\big),$$

we infer that

$$\begin{aligned}\big|h_{n+1}-h_n\big|(z) &= \big|h_n\circ\big(\mathrm{id}+w_{n+1}\big)(z) - h_n(z)\big|\\ &\leq \big|w_{n+1}(z)\big|\cdot\sup_{z'\in B_{r_{n+1}\cdot(1-3\triangle_{n+1})}}\big|h_n'(z')\big|,\end{aligned}$$

provided $z \in B_{\frac{r}{2}} \subset B_{r_{n+2}}$. But then using the Cauchy integral formula we deduce that

$$\sup_{z'\in B_{r_{n+1}\cdot(1-3\triangle_{n+1})}}\big|h_n'(z')\big| \leq C(r,d)\cdot\triangle_{n+1}^{-1} \leq C(r,d)\cdot D^n,$$

for suitable constants $C(r,d), D$. Thanks to the rapid decrease of the w_n, we then deduce that

$$\big|h_{n+1}-h_n\big|(z) \leq 2^{-(\frac{3}{2})^n},\ z\in B_{\frac{r}{2}}(0),$$

for large enough n, which implies the convergence of the h_n on $B_{\frac{r}{2}}(0)$.
Since

$$h_n^{-1}\circ f\circ h_n = \lambda\cdot z + u_n,\ z\in B_{\frac{r}{2}}(0)$$

by construction and $|u_n| \to 0$ rapidly there, we obtain that

$$h^{-1}\circ f\circ h(z) = \lambda\cdot z,\ z\in B_{\frac{r}{2}}(0),$$

as desired.