THE STABLE AND UNSTABLE MANIFOLDS CLOSE TO A HYPERBOLIC FIXED POINT: HADAMARD-PERRON

Following the account in Zehnder's book, we now investigate the question whether the stable and unstable manifold, and more precisely, their intersection with a small neighborhood around the hyperbolic fixed point, have a nice structure. This does not follow from Hartman-Grobman, which does not allow us to infer a manifold structure for these sets.

Let $\phi \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ a diffeomorphism, with *hyperbolic fixed point* $x_0 = 0$. We can then write

$$\phi(x) = Ax + g(x),\ A = D\phi(0) \in \mathrm{Mat}(n \times n;\mathbb{R}),\ Dg(0) = 0.$$

We let

$$\mathbb{R}^n = E_+ \oplus E_-$$

the decomposition into the stable and unstable space with respect to A, and we let $A_\pm = A|_{E_\pm}$. Letting $\|\cdot\|$ be the adapted norm with respect to which A_+, A_-^{-1} are contracting, we introduce the closed 'cubes'

$$Q := \{x = (x_+, x_-) \in E_+ \oplus E_-,\ \|x_\pm\| \leq r\},\ r > 0.$$

The idea is to let r be sufficiently small in the end. We now specialise the earlier definition of the 'stable and unstable manifolds' as follows:

Definition 0.1. *We define the local stable and unstable manifolds as follows:*

$$W_+^{loc}(Q) := \{x \in Q,\ \phi^j(x) \in Q\,\forall j \geq 0\},$$
$$W_-^{loc}(Q) := \{x \in Q,\ \phi^{-j}(x) \in Q\,\forall j \geq 0\}.$$

Strictly speaking there is a discrepancy between this definition and the one given earlier, since we do not explicitly demand that $\phi^j(x) \longrightarrow 0$ as $j \to +\infty$ for the local stable manifold, and similarly for the unstable one. However, this will be a consequence of the main theorem below. We note right away the trivial inclusions

$$\phi\big(W_+^{loc}(Q)\big) \subset W_+^{loc}(Q),\ \phi^{-1}\big(W_-^{loc}(Q)\big) \subset W_-^{loc}(Q).$$

We would like to give a geometric characterisation of the sets $W_\pm^{loc}(Q)$. While Hartman-Grobman characterises these as sets, we can not get any smoother structure for them this way. Nonetheless, we have

Theorem 0.2. *(Hadamard-Perron) Assuming r to be sufficiently small (depending on ϕ), there is a Lipschitz continuous function $h_+ : E_+ \longrightarrow E_-$, with $h_+(0) = 0$, such that*

$$W_+^{loc}(Q) = \{x \in Q,\ \phi^j(x) \in Q\forall j \geq 0,\ \phi^j(x) \to 0,\ j \to +\infty\}$$
$$= \{x \in Q,\ x = (x_+, h(x_+)) \in E_+ \oplus E_-\}.$$

Thus $W_+^{loc}(Q)$ is given by the graph of a Lipschitz function over $E_+ \cap Q$. An analogous result holds for $W_-^{loc}(Q)$.

Remark 0.3. One can prove more: in fact, of ϕ is a C^k-diffeomorphism, then $h_\pm$ can be chosen of regularity C^k. We do not strive for this level of precision to keep the technicalities limited.

The proof of Hadamard-Perron will follow by 'localising' a 'global version', similarly to our approach to Hartman-Grobman. In fact, let us consider now the case of a C^1-diffeomorphism ϕ of $\mathbb{R}^n$ which admits a *global decomposition*

$$\phi(x) = Ax + g(x)$$

where $\phi(0) = 0$ is a hyperbolic fixed point, $A = D\phi(0)$, and

$$g \in C_b(\mathbb{R}^n, \mathbb{R}^n),\ \|g'\|_{L^\infty} < \delta$$

for some $0 < \delta \ll 1$. Then we have

Proposition 0.4. *Under the preceding assumptions, if δ is sufficiently small (depending on A), then we have*

$$\begin{aligned} W_+^{loc}(\mathbb{R}^n) &:= \{x \in \mathbb{R}^n,\, \sup_{j\geq 0}|\phi^j(x)| < \infty\} \\ &= \{x \in \mathbb{R}^n,\, \phi^j(x) \longrightarrow 0,\, j \to +\infty\} \\ &= \{x = (x_+, h(x_+))\}, \end{aligned}$$

where $h : E_+ \longrightarrow E_-$ is a Lipschitz continuous function. with

$$h(0) = 0.$$

Moreover, we have

$$\big\|h(a)\big\| \leq \varepsilon \cdot \big\|a\big\|,$$

where $\varepsilon = \varepsilon(\delta) \ll 1$.

Proof. The strategy is again an adroit use of the Banach fixed point theorem, but the metric space where we apply it will be of more sophisticated nature than in the previous instances. The proposition asserts that for each $x_+ \in E_+$ there is a unique $x_- = h(x_+) \in E_-$ such that

$$x_+ + x_-$$

is in $W_+(\mathbb{R}^n)$. Following Zehnder, we set $x_+ = a \in E_+$, and attempt to characterise $h(a)$. Let us denote the projections

$$P_+(x_+, x_-) = x_+,\ P_-(x_+, x_-) = x_-.$$

Then from lecture 8, we know that

$$\|AP_+(x)\| \leq \alpha\|x\|,\ \|A^{-1}P_-(x)\| \leq \alpha\|x\|,$$

for some $\alpha \in [0, 1)$.

Setting

$$x_0 = a + h(a),\ a \in E_+$$

for some as yet unknown $h(a)$, we will now look at the *entire forward orbit*

$$\{x_j,\ j \geq 0,\ x_{j+1} = \phi(x_j)\}.$$

Thus we have $x_{j+1} = Ax_j + g(x_j)$. Projecting onto $E_\pm$, we obtain that

$$\begin{aligned} &P_+x_{j+1} = P_+Ax_j + P_+g(x_j) = AP_+x_j + P_+g(x_j) \\ &P_-x_{j+1} = P_-Ax_j + P_-g(x_j) = AP_-x_j + P_-g(x_j), \\ &j \geq 0. \end{aligned}$$

In order to reveal the contractive property of A^{-1} on E_-, we re-arrange the second equation:

$$\begin{aligned} &P_+x_{j+1} = AP_+x_j + P_+g(x_j) \\ &P_-x_j = A^{-1}P_-x_{j+1} - A^{-1}P_-g(x_j), \\ &j \geq 0. \end{aligned}$$

The preceding can be re-written as follows: with $a \in E_+$

$$\begin{aligned} &x_j = AP_+x_{j-1} + P_+g(x_{j-1}) + A^{-1}P_-x_{j+1} - A^{-1}P_-g(x_j),\ j \geq 1, \\ &x_0 = a + A^{-1}P_-x_1 - A^{-1}P_-g(x_0). \end{aligned} \tag{0.1}$$

The key idea now is to interpret the right hand side as a map from the sequence space $\{(x_j)\}$ into itself, and to characterise its fixed points. These then correspond to the orbits with initial condition

$$x_0 = a + A^{-1}P_-x_1 - A^{-1}P_-g(x_0).$$

If we can show that the fixed point is unique, then we can set

$$h(a) = A^{-1}P_-x_1 - A^{-1}P_-g(x_0),$$

where of course we interpret $x_0 = x_0(a), x_1 = x_1(a)$.

Of course, to make this work, we need to equip the sequence space with a norm. We shall use a family of norms, indexed by a parameter $\lambda \geq 1$. Thus set (here $\{x_j\}_{j\geq 0}$ is a sequence of points in $\mathbb{R}^n$)

$$\big\|\{x_j\}_{j\geq 0}\big\|_\lambda := \sup_{j\geq 0} \lambda^j \|x_j\|. \tag{0.2}$$

We shall also use the notation $x := \{x_j\}_{j\geq 0}$ for a sequence, and then we let

$$X_\lambda := \{x,\ \|x\|_\lambda < \infty\}.$$

One easily checks that the X_λ, equipped with these norms, are Banach spaces, and that

$$X_\lambda \subset X_\mu$$

provided $\lambda \geq \mu \geq 1$. Now write for $a \in E_+$ and $x \in X_\lambda$

$$F_a(x) := \text{right hand side of } (0.1).$$

Then we observe

Lemma 0.5. *The map F_a maps each X_λ into itself. Moreover, under the hypotheses of Hadamard-Perron, we have*

$$\big\|F_a(x) - F_a(y)\big\|_\lambda \leq \lambda(\alpha + C\delta)\big\|x - y\big\|_\lambda,$$

where $C = C(n)$ is a universal constant. In particular, if $\lambda < \alpha^{-1}$ and δ is small enough, then F_a is a contraction on X_λ.

Proof. (lemma) We note that

$$\big\|g(x) - g(y)\big\| \leq C_1\delta\big\|x - y\big\|.$$

Then we can estimate for $j \geq 1$

$$\big\|\big(F_a(x) - F_a(y)\big)_j\big\| \leq \max\{\alpha\big\|x_{j-1} - y_{j-1}\big\| + C_1\delta\big\|x_{j-1} - y_{j-1}\big\|,\ \alpha\big\|x_{j+1} - y_{j+1}\big\| + C_1\delta\big\|x_j - y_j\big\|\},$$

while for $j = 0$, we get

$$\big\|\big(F_a(x) - F_a(y)\big)_0\big\| \leq \alpha\big\|x_1 - y_1\big\| + C\delta\big\|x_0 - y_0\big\|.$$

It follows that

$$\lambda^j\big\|\big(F_a(x) - F_a(y)\big)_j\big\| \leq \lambda(\alpha + C_1\delta)\big\|x - y\big\|_\lambda,\ \forall j \geq 0.$$

Taking the supremum over $j \geq 0$, we infer that

$$\big\|F_a(x) - F_a(y)\big\|_\lambda \leq \lambda(\alpha + C_1\delta)\big\|x - y\big\|_\lambda.$$

Since $F_a(0) = (a, 0\ldots)$, this immediately implies

$$\big\|F_a\big\|_\lambda \leq \|a\| + \big\|x\big\|_\lambda < \infty,$$

so F_a maps X_λ into itself, and the lemma is proved. ☐

Continuing with the proof of the proposition, we now assume that δ is small enough such that

$$\alpha + C\delta < 1,$$

and then we restrict to $1 \leq \lambda < (\alpha + C\delta)^{-1}$. Applying the Banach fixed point theorem to $\big(X_\lambda, F_a\big)$, we infer the existence of a unique fixed point $x \in \cap X_\lambda$. In particular, this implies that the unique fixed point in

$$X_1$$

corresponding to a *bounded orbit* actually has to also be in some $X_\lambda, \lambda > 1$, implying that

$$\big\|x_j\big\| \longrightarrow 0$$

as $j \longrightarrow +\infty$. Moreover, for the starting point x_0, we have

$$h(a) := A^{-1}P_-x_1 - A^{-1}P_-g(x_0).$$

When $a = 0$, then clearly the trivial orbit $x_j = 0 \, \forall j \geq 0$ satisfies the fixed point property, so $h(0) = 0$. To see the Lipschitz continuity of h, we observe that

$$\big\|F_a(x) - F_b(x)\big\|_\lambda \leq \big\|a - b\big\|,$$

and if $F_a(x) = x, F_b(y) = y$, then

$$\begin{aligned} \big\|x - y\big\|_\lambda &\leq \big\|F_a(x) - F_a(y)\big\|_\lambda + \big\|F_a(y) - F_b(y)\big\|_\lambda \\ &\leq \lambda(\alpha + C\delta)\big\|x - y\big\|_\lambda + \big\|a - b\big\| \end{aligned}$$

Setting $\lambda = 1$, and assuming that $\mu := \alpha + C\delta < 1$, we deduce that

$$\big\|x - y\big\|_1 \leq \frac{1}{1-\mu}\big\|a - b\big\|.$$

In particular, this implies that

$$\big\|h(a) - h(b)\big\| \leq \frac{1}{1-\mu}\big\|a - b\big\|.$$

To deduce the last assertion of the proposition, we now keep a fixed and vary the function g, subject to satisfying the bounds of the proposition. Then if $g_{1,2}$ satisfy the bounds of the proposition, and $x^{(1),(2)}$ are the corresponding bounded forward orbits starting at

$$a + h_{1,2}(a),$$

then we have (letting $F_a^{1,2}$ the maps with underlying function $g_{1,2}$, respectively)

$$\begin{aligned} \big\|x^{(1)} - x^{(2)}\big\|_1 &= \big\|F_a^1(x^{(1)}) - F_a^2(x^{(2)})\big\|_1 \\ &\leq \big\|F_a^1(x^{(1)}) - F_a^1(x^{(2)})\big\|_1 + \big\|F_a^1(x^{(2)}) - F_a^2(x^{(2)})\big\|_1 \\ &\leq (\alpha + C\delta)\big\|x^{(1)} - x^{(2)}\big\|_1 + \big\|F_a^1(x^{(2)}) - F_a^2(x^{(2)})\big\|_1. \end{aligned}$$

Then we note that

$$\big\|F_a^1(x^{(2)}) - F_a^2(x^{(2)})\big\|_1 \leq \sup_{j\geq 0}\big\|g_1(x_j^{(2)}) - g_2(x_j^{(2)})\big\|.$$

Now we set $g_2 = g$, $g_1 = 0$, whence

$$x^{(1)}(a) = \{a, Aa, A^2a, \ldots\},$$

and further

$$\sup_{j\geq 0}\big\|g_1(x_j^{(2)}) - g_2(x_j^{(2)})\big\| \leq C_1\delta\big\|x^{(2)}\big\|_1 \leq C_2\delta\big\|a\big\|.$$

It follows that

$$\big\|x^{(1)} - x^{(2)}\big\|_1 \leq \frac{C_2\delta}{1-\mu}\big\|a\big\|, \, \mu = \alpha + C\delta.$$

In particular, since

$$x_0^{(1)} - x_0^{(2)} = -h(a),$$

we deduce that

$$\big\|h(a)\big\| \leq \frac{C_2\delta}{1-\mu}\big\|a\big\|.$$

□

We can now give the proof of Hadamard-Perron:

Proof. (Hadamard-Perron) Given $r > 0$ small, we first modify the diffeomorphism $\phi(x) = Ax + g(x)$ so that the preceding proposition can be applied to it. For this, let $\chi \in C_0^\infty(\mathbb{R}^n)$ a smooth function with the property that

$$\chi\big|_{|x|\leq 1} = 1, \, \mathrm{supp}(\chi) \subset B_2(0),$$

and consider

$$\widetilde{\phi}(x) := Ax + \chi\big(\frac{x}{r}\big) \cdot g(x) =: Ax + \widetilde{g}(x).$$

Then observe that

$$\begin{aligned}\big|\nabla_x\widetilde{g}\big| &\leq Cr^{-1}\cdot\big\|g\big\|_{L^\infty(B_{2r}(0))}+\big\|\nabla g\big\|_{L^\infty(B_{2r}(0))}\\ &\leq \big(\widetilde{C}r^{-1}\cdot r+1\big)\big\|\nabla g\big\|_{L^\infty(B_{2r}(0))}\end{aligned}$$

We conclude that by choosing r small enough, the hypotheses of the proposition with $\widetilde{g}$ replacing g are satisfied. In fact, the map $\widetilde{\phi}$ is a homeomorphism of $\mathbb{R}^n$ according to Lemma 4.1 from lecture 8 if r is small enough, and its inverse is C^1.

Pick $0<r_1\leq r$ small enough, such that the 'boxed neighborhood'

$$Q:=\{x=x_++x_-,\,\|x_+\|\leq r_1,\,\|x_-\|\leq r_1\}\subset B_r(0).$$

Also, let

$$\widetilde{W}_+^{loc}(\mathbb{R}^n)=\{x\in\mathbb{R}^n,\,\sup_{j\geq 0}\|\widetilde{\phi}^j(x)\|<\infty\}.$$

According to the preceding proposition, we can write

$$\widetilde{W}_+^{loc}(\mathbb{R}^n)=\{x_++h(x_+),\,x_+\in E_+\},$$

for Lipschitz continuous h as in the proposition.

We now claim that if

$$W_+^{loc}(Q)=\{x\in Q,\,\phi^j(x)\in Q\,\forall j\geq 0\},$$

then we have

$$W_+^{loc}(Q)=\widetilde{W}_+^{loc}(\mathbb{R}^n)\cap Q \tag{0.3}$$

for $r_1>0$ small enough.

(i) $W_+^{loc}(Q)\subset\widetilde{W}_+^{loc}(\mathbb{R}^n)\cap Q$. If $x\in W_+^{loc}(Q)$, then

$$\widetilde{\phi}^j(x)=\phi^j(x)\in Q\,\forall j\geq 0,$$

and hence

$$x\in\widetilde{W}_+^{loc}(\mathbb{R}^n)\cap Q.$$

(ii) $\widetilde{W}_+^{loc}(\mathbb{R}^n)\cap Q\subset W_+^{loc}(Q)$. Here we show that if

$$x\in\widetilde{W}_+^{loc}(\mathbb{R}^n)\cap Q,$$

then

$$\widetilde{\phi}(x)\in\widetilde{W}_+^{loc}(\mathbb{R}^n)\cap Q.$$

This then implies inductively that

$$\widetilde{\phi}^j(x)\in Q,\,\forall j\geq 0,$$

and thence

$$\phi^j(x)\in Q\,\forall j\geq 0.$$

Now if

$$x\in\widetilde{W}_+^{loc}(\mathbb{R}^n)\cap Q,$$

then the proposition implies that

$$x=x_++h(x_+),\,\|x_+\|\leq r_1.$$

Applying $\widetilde{\phi}=A\cdot\mathrm{id}+\widetilde{g}$, we infer that

$$\widetilde{\phi}(x)=Ax_++Ah(x_+)+\widetilde{g}(x).$$

Then we estimate

$$\big\|P_+\widetilde{\phi}(x)\big\|\leq\big\|Ax_+\big\|+\big\|P_+Ah(x_+)\big\|+\big\|P_+\widetilde{g}(x)\big\|\leq\alpha r_1+C_1\varepsilon(\delta)r_1+C_2\delta r_1,$$

where $\delta = \delta(r)$ can be made arbitrarily small by choosing $r > 0$ small enough, and $\lim_{\delta \to 0} \varepsilon(\delta) = 0$. But since $\alpha < 1$, by choosing r small enough, we can ensure that

$$\alpha r_1 + C_1 \varepsilon(\delta) r_1 + C_2 \delta r_1 \leq r_1.$$

One proceeds similarly for $P_- \widetilde{\phi}(x)$.

We have now shown (0.3), which in conjunction with the proposition implies Hadamard-Perron. $\square$