

Math of ML : Exercises 4 *

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First some warm-up exercises on the definitions of positive definite kernel and of RKHS.

Exercise 1 (Cauchy-Schwarz inequality). *Let $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ be a positive definite kernel (recall Definition 3.1 in Lecture 4). Prove that for any $x, y \in \mathcal{X}$ we have $k(x, y)^2 \leq k(x, x)k(y, y)$. Deduce that $\sup_{x, y} k(x, y) = \sup_x k(x, x)$ (it may be $+\infty$).*

Exercise 2 (Bounded kernels). *Let $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ be a positive definite kernel. Suppose that there exists some constant $b > 0$ such that for any $x \in \mathcal{X}$ we have $k(x, x) \leq b^2$. Let \mathcal{H} be the RKHS associated to the kernel k and let $f \in \{g \in \mathcal{H} : \|g\|_{\mathcal{H}} \leq 1\}$ be a function in the unit ball of \mathcal{H} (recall Definition 3.3 in Lecture 4). Prove that $\sup_{x \in \mathcal{X}} |f(x)| \leq b$.*

The next exercise lets you construct kernels from other kernels. It can be instructive to also ask yourself what the corresponding RKHS's are (there is not always a simple answer).

Exercise 3 (Calculus of positive definite kernels). *Let $k_1, k_2 : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ be two positive definite kernels. Show that the following functions k are also positive definite kernels.*

1. $k(x, y) = k_1(x, y) + k_2(x, y)$.
2. $k(x, y) = ak_1(x, y)$, where $a > 0$.
3. $k(x, y) = k_1(x, y)k_2(x, y)$.
4. $k(x, y) = p(k_1(x, y))$, where p is a polynomial with positive coefficients.
5. $k(x, y) = \lim_{l \rightarrow \infty} k_l(x, y)$ (provided that the limit exists), where k_1, k_2, k_3, \dots is a sequence of positive definite kernels over \mathcal{X} .
6. $k(x, y) = \exp(k_1(x, y))$.
7. $k(x, y) = k_1(x, y) / \sqrt{k_1(x, x)k_1(y, y)}$ (you may assume $k_1(x, x) > 0$ for any $x \in \mathcal{X}$).
8. $k(x, y) = f(x) \cdot f(y)$ where $f : \mathcal{X} \rightarrow \mathbb{R}$ is any function.
9. $k(x, y) = h(\psi(x), \psi(y))$, where $h : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$ is a p.d. kernel and $\psi : \mathcal{X} \rightarrow \mathcal{Y}$ is arbitrary.

Which of the following functions are positive definite kernels over $\mathcal{X} = \mathbb{R}^d$?

- $k(x, y) = \|x - y\|$.
- $k(x, y) = \|x - y\|^2$.
- $k(x, y) = x^\top B y$, where $B \in \mathbb{R}^{d \times d}$ is a symmetric positive-semi-definite matrix.
- $k(x, y) = \exp(-\frac{\|x - y\|^2}{2\sigma^2})$, where $\sigma > 0$.

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In this next exercise you characterize all translation-invariant kernels on the 1-D torus. (Translation-invariant kernels on \mathbb{R}^d will be discussed in the next lecture.)

Exercise 4 (Translation-invariant kernels on $[0, 1]$ [Bach, 2024, Sec. 7.3.2]).

1. Let q be a continuous 1-periodic function and $k(x, y) = q(x - y)$. Show that k is a positive definite kernel on $[0, 1]$ if and only if all of the Fourier coefficients of q are non-negative. Express the associated RKHS in terms of the Fourier coefficients of q .

Hint/Reminder: The Fourier series decomposition of a function $f \in L_2([0, 1])$ is $f(x) = \sum_{m \in \mathbb{Z}} e^{2im\pi x} \hat{f}_m$, and the Fourier coefficients are given by $\hat{f}_m = \int_0^1 f(x) e^{-2im\pi x} dx \in \mathbb{C}$.

2. Using question 1., show that the set $\left\{ f : [0, 1] \rightarrow \mathbb{R} \text{ s.t. } \left(\int_0^1 f \right)^2 + \int_0^1 |f'(x)|^2 dx < \infty \right\}$ equipped with the inner product

$$\langle f, g \rangle = \left(\int_0^1 f \right) \left(\int_0^1 g \right) + \frac{1}{(2\pi)^2} \int_0^1 f'(x)g'(x)dx$$

is a RKHS and identify the corresponding kernel function (you may leave it in the form of a trigonometric series, $\sum_n A_n \cos(nX) + B_n \sin(nX)$).

References

Francis Bach. *Learning theory from first principles*. MIT press, 2024. URL https://www.di.ens.fr/~fbach/ltfp_book.pdf.