Ergodic Theory
Solutions to Problem Sheet 9

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P1. Let $(X, \mathscr{B}, \mu, T)$ be a measure preserving system and let $f \in L^2(X)$. Let μ_f be the spectral measure of f with respect to U_T, and let $P_T f$ be the orthogonal projection onto the closed subspace of T-invariant functions in $L^2(X, \mu)$. Show that:

(a) $\mu_f(\{0\}) = \|P_T f\|_2^2$.

(b) If the system is ergodic, then: $\mu_f(\{0\}) = 0$ if and only if $\int f \, d\mu = 0$.

(c) $\mu_f(\mathbb{T}) = ||f||_2^2$.

(a) We want to compute

$$\mu_f(\{0\}) = \int_{\mathbb{T}} \mathbb{1}_{\{0\}}(x) \, d\mu_f(x).$$

Notice that

$$\mathbb{1}_{\{0\}}(x) = \operatorname*{lím}_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(nx),$$

and by the Dominated convergence theorem, we get

$$\mu_f(\{0\}) = \operatorname*{lím}_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_{\mathbb{T}} e(nx) \, d\mu_f(x) = \operatorname*{lím}_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle T^n f, f \rangle.$$

Now, write $f = P_T f + f_{\text{erg}}$ where f_{erg} is the coboundary part of f. Using the mean ergodic theorem and orthogonality, we get that

$$\begin{aligned}\mu_f(\{0\}) &= \operatorname*{lím}_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle T^n P_T f, P_T f \rangle + \langle T^n f_{\text{erg}}, P_T f \rangle + \langle T^n P_T f, f_{\text{erg}} \rangle + \langle T^n f_{\text{erg}}, f_{\text{erg}} \rangle \\ &= \operatorname*{lím}_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle P_T f, P_T f \rangle + \langle T^n f_{\text{erg}}, f_{\text{erg}} \rangle \\ &= \langle P_T f, P_T f \rangle = \|P_T f\|_2^2.\end{aligned}$$

$\square$

(b) Recall the mean ergodic theorem for the ergodic case. If T is ergodic, then $P_T f = \int f \, d\mu$. Thus

$$\mu_f(\{0\}) = \left\| \int f \, d\mu \right\|_2^2 = \left(\int f \, d\mu \right)^2,$$

from which the equivalence follows. ☐

(c) Using the spectral theorem we have that

$$\mu(\mathbb{T}) = \int_{\mathbb{T}} d\mu_f = \langle U_T^0 f, f\rangle = \langle f, f\rangle = \|f\|_2^2.$$

☐

In the following 3 exercises, $\mathcal{H}$ is a separable Hilbert space, and for any $w \in \mathcal{H}$, we denote by $\mathcal{H}_w$ the subspace generated by w, and we define $\chi_n(x) = e^{2\pi inx}$. We define a unitary operator $U : \mathcal{H} \mapsto \mathcal{H}$. Recall that the spectral measure μ_w of w with respect to U is a finite Borel measure on $\mathbb{T} = \mathbb{R} \setminus \mathbb{Z}$, uniquely determined by the following property

$$\langle U^n w, w\rangle = \int_{\mathbb{T}} \chi_n(x) \mathrm{d}\mu_w(x) \quad \forall n \in \mathbb{Z}. \tag{1}$$

Additionally, one can show that using this measure we obtain a unitary isomorphism $\mathcal{H}_w \cong L^2_{\mu_w}(\mathbb{T})$ where the action of U on $\mathcal{H}_w$ corresponds to the action of M_{χ_1} on $L^2_{\mu_w}(\mathbb{T})$ and w corresponds to $\mathbb{1} \in L^2_{\mu_w}(\mathbb{T})$, and where M_{χ_1} is the unitary operator given by $M_{\chi_1} : f \in L^2_{\mu_w}(\mathbb{T}) \mapsto \chi_1 f \in L^2_{\mu_w}(\mathbb{T})$.

P2. Let $w, z \in \mathcal{H}$. Show that there exists a complex signed measure $\mu_{w,z}$ such that

$$\langle U^n w, z\rangle = \int_{\mathbb{T}} \chi_n \mathrm{d}\mu_{w,z} \quad \forall n \in \mathbb{Z}.$$

Recalling the polarization identity: $\langle w, z\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k ||w + i^k z||^2$, one can identify a suitable candidate for the sought-after measure $\mu_{w,z}$. Set $\mu_{w,z} = \frac{1}{4}\sum_{k=0}^{3} i^k \mu_{w+i^k z}$.

By the spectral theorem, for $n \in \mathbb{Z}$ we obtain

$$\int_{\mathbb{T}} \chi_n \mathrm{d}\mu_{w,z} = \frac{1}{4}\sum_{k=0}^{3} i^k \int_{\mathbb{T}} \chi_n \mathrm{d}\mu_{w+i^k z} = \frac{1}{4}\sum_{k=0}^{3} i^k \langle U^n(w + i^k z), w + i^k z\rangle.$$

Next, we observe that for any k we have the following expression

$$\langle U^n(w + i^k z), w + i^k z\rangle = \langle U^n w, w\rangle + i^k\langle U^n z, w\rangle + i^{-k}\langle U^n w, z\rangle + \langle U^n z, z\rangle.$$

Note that $\sum_{k=0}^{3} i^k = \sum_{k=0}^{3} i^{2k} = 0$. Therefore, the first, second and fourth term drop out and we obtain

$$\int_{\mathbb{T}} \chi_n \mathrm{d}\mu_{w,z} = \frac{1}{4}\sum_{k=0}^{3}\langle U^n w, z\rangle = \langle U^n w, z\rangle,$$

as desired. ☐

P3. Let $\lambda \in \mathbb{T}$, $0 \neq w \in \mathcal{H}$. Show that w is an eigenvector of U for eigenvalue $\chi_1(\lambda)$ if and only if $\mu_w = ||w||^2\delta_\lambda$, where δ_λ is the Dirac measure at the point λ.

> For the forward direction, assume that w is an eigenvector of U with eigenvalue $\chi_1(\lambda)$. By the unitary isomorphism, this implies that $\mathbb{1}$ is an eigenvector of M_{χ_1} with eigenvalue λ, which gives that
>
> $$M_{\chi_1}^n \mathbb{1} = \lambda^n \mathbb{1}.$$
>
> Note that $\chi_1^n = \chi_n$, so $M_{\chi_1}^n \mathbb{1} = \chi_n \mathbb{1}$, and thus μ_w-a.e., $\lambda^n = \chi_n$ holds. This implies that
>
> $$\int_{\mathbb{T}} \chi_n \mathrm{d}\mu_w = \lambda^n \int_{\mathbb{T}} \mathrm{d}\mu_w = \lambda^n ||w||^2.$$
>
> Define the measure $\nu := ||w||^2 \delta_\lambda$, from which we observe that $\lambda^n ||w||^2 = \int_{\mathbb{T}} \chi_n \mathrm{d}\nu$. Recalling that every character of $\mathbb{T}$ is of the form χ_n, we have shown that the measures ν and μ_w agree on characters, which implies that they agree on trigonometric polynomials. By Stone-Weierstrass they agree on $C(\mathbb{T})$. By density, they agree on $L^2(\mathbb{T})$, which contains characters, and so $\mu_w = ||w||^2 \delta_\lambda$, as desired.
>
> For the other direction, we assume that $\mu_w = ||w||^2 \delta_\lambda$ for $\lambda \in \mathbb{T}$. Since we have a point-mass measure, this implies that μ_w-a.e., $\chi_n = \chi_n(\lambda)$ for all n. In particular, $\chi_1 = \chi_1(\lambda)$ holds μ_w-a.e., which shows that $\mathbb{1}$ is an eigenvector of M_{χ_1} with eigenvalue $\chi_1(\lambda)$. By the unitary isomorphism $\mathcal{H}_w \cong L^2_{\mu_w}(\mathbb{T})$, this implies that w is an eigenvector of U with eigenvalue $\chi_1(\lambda)$, as claimed. □

P4. In this exercise, we give a slightly different proof of Wiener's lemma.

Let μ be a finite Borel measure on $\mathbb{T}$ and $p_n(\mu) := \int_{\mathbb{T}} \chi_n \mathrm{d}\mu$.

(a) Show that μ has at most countably many atoms which we denote by $x_1, x_2, \ldots$.

> This follows directly from μ being a finite measure as this implies that for any $\delta > 0$ the set of atoms x with $\mu(\{x\}) > \delta$ is finite, which in turn gives the claim (by applying this with $\delta = \frac{1}{n}$). □

(b) We define the function $D_N \in C(\mathbb{T})$ by

$$D_N = \sum_{n=-N}^{N} \chi_n.$$

Show that D_N is real-valued and that it can also be written as

$$D_N(x) = \begin{cases} 2N+1 & \text{if } x = 0 \in \mathbb{T} \\ \dfrac{\sin\left((N+\frac{1}{2})2\pi x\right)}{\sin(\pi x)} & \text{if } x \neq 0 \end{cases}$$

Moreover, show that it satisfies

$$\int_{\mathbb{T}} D_N(x) \mathrm{d}x = 1.$$

The case $x = 0$ follows immediately from the definition of D_N.

For $x \neq 0$, we recognise that D_N is a geometric series and using the standard relation $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ we obtain our claim

$$\begin{aligned} D_N(x) = \sum_{n=-N}^{N} (e^{2\pi i x})^n &= e^{-2\pi i N x}\big(1 + \cdots + (e^{2\pi i x})^{2N}\big) \\ &= e^{-2\pi i N x}\frac{(e^{2\pi i x})^{2N+1} - 1}{e^{2\pi i x} - 1} \\ &= \frac{e^{2\pi i (N+\frac{1}{2})x} - e^{-2\pi i (N+\frac{1}{2})x}}{e^{\pi i x} - e^{-\pi i x}} \\ &= \frac{\sin\big((N+\frac{1}{2})2\pi x\big)}{\sin(\pi x)}. \end{aligned}$$

Finally, for the integral we proceed as follows

$$\int_{\mathbb{T}} D_N(x)\mathrm{d}x = \int_{\mathbb{T}} \sum_{n=-N}^{N} \chi_n(x)\mathrm{d}x = \sum_{n=-N}^{N} \int_{\mathbb{T}} \chi_n(x)\mathrm{d}x = \sum_{n=-N}^{N} \mathbb{1}_{\{0\}}(n) = 1.$$

☐

(c) Show that any finite measure ν on $\mathbb{T}$ satisfies the following

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} p_n(\nu) = \nu(\{0\}). \tag{2}$$

We have

$$\frac{1}{2N+1} \sum_{n=-N}^{N} p_n(\nu) = \frac{1}{2N+1} \sum_{n=-N}^{N} \int_{\mathbb{T}} \chi_n \mathrm{d}\nu = \int_{\mathbb{T}} \frac{1}{2N+1} D_N \mathrm{d}\nu.$$

By part **(b)**, we know that $\dfrac{1}{2N+1} D_N$ converges to 0 for $x \neq 0$ as $N \to \infty$, it is bounded by 1 and is equal to 1 for $x = 0$. Therefore, by the Dominated Convergence Theorem, we obtain

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} p_n(\nu) = \int_{\mathbb{T}} \mathbb{1}_{\{0\}} \mathrm{d}\nu = \nu(\{0\}).$$

☐

(d) Denote by $\triangle = \{(t,t) \,|\, t \in \mathbb{T}\}$. Let ν be the push-forward of the product measure $\mu \times \mu$ on $\mathbb{T}^2$ under the map $(t_1, t_2) \in \mathbb{T}^2 \mapsto t_1 - t_2 \in \mathbb{T}$. Show that $\nu(\{0\}) = \mu \times \mu(\triangle)$ and $p_n(\nu) = |p_n(\mu)|^2$.

By definition, for any measurable set $A \subset \mathbb{T}$, we have $\nu(A) = \mu \times \mu(D^{-1}(A))$, where D is the difference map $D : (t_1, t_2) \in \mathbb{T}^2 \mapsto t_1 - t_2 \in \mathbb{T}$. Observe that $D^{-1}(\{0\}) = \triangle$, so the first claim follows immediately.

For the second claim, note that for any simple function $f = \mathbb{1}_A$ for $A \subset \mathbb{T}$ measurable, the following identity is satisfied:

$$\int_{\mathbb{T}} f \, \mathrm{d}\nu = \int_{\mathbb{T}^2} f \circ D \, \mathrm{d}(\mu \times \mu).$$

Therefore, this identity also holds for any linear combinations of simple functions, and therefore for any L^1-function, which follows by density. In particular, we can take $f = \chi_n$ and using Fubini's Theorem, we conclude

$$\begin{aligned} p_n(\nu) = \int_{\mathbb{T}} \chi_n \mathrm{d}\nu &= \int_{\mathbb{T}^2} \chi_n \circ D \, \mathrm{d}(\mu \times \mu) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} \chi_n(t_1 - t_2) \mathrm{d}\mu(t_1) \mathrm{d}\mu(t_2) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} \chi_n(t_1) \overline{\chi_n(t_2)} \mathrm{d}\mu(t_1) \mathrm{d}\mu(t_2) \\ &= p_n(\mu) \overline{p_n(\mu)} \\ &= |p_n(\mu)|^2. \end{aligned}$$

□

(e) Show that $\mu \times \mu(\triangle) = \sum_{i=1}^{\infty} \mu(\{x_i\})^2$ and deduce Wiener's lemma:

$$\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |p_n(\mu)|^2 = \sum_{i=1}^{\infty} \mu(\{x_i\})^2. \tag{3}$$

Using Fubini's Theorem once again, we have

$$\mu \times \mu(\triangle) = \int_{\mathbb{T}} \int_{\mathbb{T}} \mathbb{1}_{\triangle}(t_1, t_2) \mathrm{d}\mu(t_1) \mathrm{d}\mu(t_2) = \int_{\mathbb{T}} \mu(\{t_2\}) \mathrm{d}\mu(t_2).$$

By part **(a)**, the last integrand is non-zero on the countable set of atoms we identified, and so we obtain

$$\begin{aligned} \int_{\mathbb{T}} \mu(\{t_2\}) \mathrm{d}\mu(t_2) &= \int_{\mathbb{T}} \sum_{i=1}^{\infty} \mathbb{1}_{\{x_i\}}(t_2) \mu(\{t_2\}) \mathrm{d}\mu(t_2) \\ &= \sum_{i=1}^{\infty} \mu(\{x_i\})^2, \end{aligned}$$

as desired. Finally, using parts **(c)** and **(d)**, we deduce Wiener's lemma

$$\begin{aligned} \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |p_n(\mu)|^2 &= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} p_n(\nu) \\ &= \nu(\{0\}) \\ &= \mu \times \mu(\triangle) \\ &= \sum_{i=1}^{\infty} \mu(\{x_i\})^2. \end{aligned}$$

□