Ergodic Theory
Problem Sheet 9

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P1. Let $(X, \mathscr{B}, \mu, T)$ be a measure preserving system and let $f \in L^2(X)$. Let μ_f be the spectral measure of f with respect to U_T, and let $P_T f$ be the orthogonal projection onto the closed subspace of T-invariant functions in $L^2(X)$. Show that:

- **(a)** $\mu_f(\{0\}) = \|P_T f\|_2^2$.
- **(b)** If the system is ergodic, then: $\mu_f(\{0\}) = 0$ if and only if $\int f \, d\mu = 0$.
- **(c)** $\mu_f(\mathbb{T}) = ||f||_2^2$.

In the following 3 exercises, $\mathcal{H}$ is a separable Hilbert space, and for any $w \in \mathcal{H}$, we denote by $\mathcal{H}_w$ the subspace generated by w, and we define $\chi_n(x) = e^{2\pi i n x}$. We define a unitary operator $U : \mathcal{H} \mapsto \mathcal{H}$. Recall that the spectral measure μ_w of w with respect to U is a finite Borel measure on $\mathbb{T} = \mathbb{R} \setminus \mathbb{Z}$, uniquely determined by the following property

$$\langle U^n w, w \rangle = \int_{\mathbb{T}} \chi_n(x) \mathrm{d}\mu_w(x) \quad \forall n \in \mathbb{Z}. \tag{1}$$

Additionally, one can show that using this measure we obtain a unitary isomorphism $\mathcal{H}_w \cong L^2_{\mu_w}(\mathbb{T})$ where the action of U on $\mathcal{H}_w$ corresponds to the action of M_{χ_1} on $L^2_{\mu_w}(\mathbb{T})$ and w corresponds to $\mathbb{1} \in L^2_{\mu_w}(\mathbb{T})$, and where M_{χ_1} is the unitary operator given by $M_{\chi_1} : f \in L^2_{\mu_w}(\mathbb{T}) \mapsto \chi_1 f \in L^2_{\mu_w}(\mathbb{T})$.

P2. Let $w, z \in \mathcal{H}$. Show that there exists a complex signed measure $\mu_{w,z}$ such that

$$\langle U^n w, z \rangle = \int_{\mathbb{T}} \chi_n \mathrm{d}\mu_{w,z} \quad \forall n \in \mathbb{Z}.$$

P3. Let $\lambda \in \mathbb{T}$, $0 \neq w \in \mathcal{H}$. Show that w is an eigenvector of U for eigenvalue $\chi_1(\lambda)$ if and only if $\mu_w = ||w||^2 \delta_\lambda$, where δ_λ is the Dirac measure at the point λ.

P4. In this exercise, we give a slightly different proof of Wiener's lemma.

Let μ be a finite Borel measure on $\mathbb{T}$ and $p_n(\mu) := \int_{\mathbb{T}} \chi_n \mathrm{d}\mu$.

- **(a)** Show that μ has at most countably many atoms which we denote by $x_1, x_2, \ldots$.

- **(b)** We define the function $D_N \in C(\mathbb{T})$ by

$$D_N = \sum_{n=-N}^{N} \chi_n.$$

Show that D_N is real-valued and that it can also be written as

$$D_N(x) = \begin{cases} 2N+1 & \text{if } x = 0 \in \mathbb{T} \\ \dfrac{\sin\left((N+\frac{1}{2})2\pi x\right)}{\sin(\pi x)} & \text{if } x \neq 0 \end{cases}$$

Moreover, show that it satisfies

$$\int_{\mathbb{T}} D_N(x)\mathrm{d}x = 1.$$

(c) Show that any finite measure ν on $\mathbb{T}$ satisfies the following

$$\text{lím}_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} p_n(\nu) = \nu(\{0\}). \tag{2}$$

(d) Denote by $\triangle = \{(t,t) \,|\, t \in \mathbb{T}\}$. Let ν be the push-forward of the product measure $\mu \times \mu$ on $\mathbb{T}^2$ under the map $(t_1, t_2) \mapsto t_1 - t_2$. Show that $\nu(\{0\}) = \mu \times \mu(\triangle)$ and $p_n(\nu) = |p_n(\mu)|^2$.

(e) Show that $\mu \times \mu(\triangle) = \sum_{i=1}^{\infty} \mu(\{x_i\})^2$ and deduce Wiener's lemma:

$$\text{lím}_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |p_n(\mu)|^2 = \sum_{i=1}^{\infty} \mu(\{x_i\})^2. \tag{3}$$