Ergodic Theory
Solutions to Problem Sheet 8

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P1. In this exercise, we identify the Pontryagin dual of $\mathbb{R}^n$.

(a) We define for $a \in \mathbb{R}^n$ the map $\chi_a : x \in \mathbb{R}^n \mapsto e^{2\pi i x\cdot a}$. Prove that any character $\chi \in \widehat{\mathbb{R}^n}$ is of the form χ_a for some $a \in \mathbb{R}^n$.
[**Hint:** Show that χ is smooth and find an ODE that is satisfied by χ.]

It is straightforward to check that χ_a is a character for all $a \in \mathbb{R}^n$.
Let $n = 1$ first, and let χ be an arbitrary character. Notice that, by substitution, the following holds:

$$\forall \delta > 0, \forall s \in \mathbb{R} \quad \int_s^{s+\delta} \chi(x)\mathrm{d}x = \int_0^{\delta} \chi(s+y)\mathrm{d}y = \chi(s)\int_0^{\delta} \chi(y)\mathrm{d}y \tag{1}$$

By standard calculus, we know that

$$\int_s^{s+\delta} \chi(x)\mathrm{d}x = \int_0^{s+\delta} \chi(x)\mathrm{d}x - \int_0^{s} \chi(x)\mathrm{d}x \tag{2}$$

is a continuously differentiable function of s. Since χ is continuous and $\chi(0) = 1$, we can choose $\delta > 0$ such that $\int_0^{\delta} \chi(x)\mathrm{d}x \neq 0$. Combining (1) and (2), we see that χ is continuously differentiable as a function of s.

By definition, for any $s, t \in \mathbb{R}$, $\chi(s+t) = \chi(s)\chi(t)$. Differentiating with respect to s, we obtain the differential equation

$$\chi'(s+t) = \chi'(s)\chi(t),$$

which, in particular for $s = 0$, gives the ODE

$$\chi'(t) = \chi'(0)\chi(t). \tag{3}$$

We recognise (3) as a very simple ODE for which we know that the unique solution is given by $\chi(t) = e^{\chi'(0)t}$.

In particular χ is smooth and since $|\chi(t)| = 1$, we obtain

$$1 = |\chi(1)| = |e^{\chi'(0)}| = e^{\mathrm{Re}(\chi'(0))}.$$

This implies that $\text{Re}(\chi'(0)) = 0$, so there must exist an $a \in \mathbb{R}$ such that $\chi'(0) = 2\pi i a$, which is the desired form for χ.

For $n > 1$, we write $t \in \mathbb{R}^n$ as $t = \sum_{i=1}^n t_i e_i$, where e_i, for $i = 1, \ldots, n$, denote the standard basis vectors. Therefore, we obtain

$$\chi(t) = \chi\Big(\sum_{i=1}^n t_i e_i\Big) = \prod_{i=1}^n \chi(t_i e_i). \tag{4}$$

By the one-dimensional case, we know that the RHS of (4) is smooth, and therefore χ is smooth as a function of t. Moreover, we also know that for $i = 1, \ldots, n$, there is an $a_i \in \mathbb{R}$ so that $\chi(t_i e_i) = e^{2\pi i a_i t_i}$. Finally, for $a = (a_1, \ldots, a_n)^{\mathrm{T}} \in \mathbb{R}^n$, we obtain that

$$\chi(t) = \prod_{i=1}^n e^{2\pi i a_i t_i} = e^{2\pi i a \cdot t}$$

as desired. □

(b) Prove that the map $a \in \mathbb{R}^n \mapsto \chi_a \in \widehat{\mathbb{R}^n}$ is a homeomorphism. This proves that the Pontryagin dual of $\mathbb{R}^n$ is $\mathbb{R}^n$.

By part (a), we know that the given map is surjective. Injectivity follows by the following argument.

Let $a_1, a_2 \in \mathbb{R}^n$ such that $\chi_{a_1} = \chi_{a_2}$. Then, we observe that for $a = a_1 - a_2$ we have $\chi_a = 1$, which further implies that $x \cdot a \in \mathbb{Z}$ for every $x \in \mathbb{R}^n$. Recall that connectedness is preserved under continuous maps, and in our case $\mathbb{R}^n$ is connected and the map $x \mapsto x \cdot a$ is continuous. Observe that $\mathbb{Z}$ is a discrete space, and in the discrete topology, no space with more than one point is connected. Therefore, the map $x \mapsto x \cdot a$ must be constant, and in particular, $x \cdot a = 0$, $\forall x \in \mathbb{R}^n$, which implies that $a = 0$ and $a_1 = a_2$. This shows injectivity.

It only remains to prove that the map $a \in \mathbb{R}^n \mapsto \chi_a \in \widehat{\mathbb{R}^n}$ is continuous. For this, let $\chi_a \in \widehat{\mathbb{R}^n}$, $\epsilon > 0$ and $K \subset \mathbb{R}^n$ compact be given. To show that the map is continuous, we need to show that there is $\delta > 0$ so that

$$||s - a|| < \delta \implies \chi_s \in B_{K,\epsilon}(\chi_a),$$

where $B_{K,\epsilon}(\chi) = \{\psi \in \widehat{\mathbb{R}^n} \,|\, ||\psi - \chi||_{K,\infty} < \epsilon\}$ and $||\psi - \chi||_{K,\infty} := \sup_{x \in K} |\psi(x) - \chi(x)|$.
To that end, we first notice that the map $\alpha \in \mathbb{R} \mapsto e^{2\pi i \alpha}$ is continuous at 0. Therefore, let $\delta_1 > 0$ be such that $|e^{2\pi i \alpha} - 1| < \epsilon$ for $|\alpha| < \delta_1$. Using compactness and the Cauchy-Schwarz inequality, we obtain that for any $x \in K$

$$|x \cdot (a - s)| \leq ||x||\, ||a - s|| \leq c||a - s||$$

for $c = \sup_{x \in K} ||x||$. Next, choose $\delta > 0$ so that $c\delta < \delta_1$. Therefore, if $||s - a|| < \delta$, we obtain

$$||\chi_a - \chi_s||_{K,\infty} = \sup_{x\in K} |\chi_a(x) - \chi_s(x)| = \sup_{x\in K} |\chi_{a-s}(x) - 1|$$
$$= \sup_{x\in K} |e^{2\pi i x\cdot(a-s)} - 1| < \epsilon,$$

as desired.

The continuity of the inverse map follows. Namely, we know that for a locally compact abelian group G, the Pontryagin dual $\widehat{G}$, when endowed with the topology given by uniform convergence on compacts, is also a locally compact abelian topological group. Since $\mathbb{R}^n$ is locally compact, σ-compact and Hausdorff, $\widehat{\mathbb{R}^n}$ is locally compact, and the map $a \in \mathbb{R}^n \mapsto \chi_a \in \widehat{\mathbb{R}^n}$ is a continuous bijection, it follows by the Open Mapping Theorem that the map is open, and therefore it is a homeomorphism. □

P2. Show that given any countable subgroup $K \leq \mathbb{S}^1$, there exists a measure preserving system $(X, \mathcal{B}, \mu, T)$ on a Borel probability space such that K is the point-spectrum of T.

[**Hint:** To construct the system take $X = \widehat{K}$, $\mu = m_X$ the normalized Haar measure on X, and T to be some appropriate group rotation.]

Give K the discrete topology, so that the dual group $\widehat{K}$ is a compact metric abelian group. Following the hint, we take $X = \widehat{K}$, $\mu = m_X$ the normalized Haar measure on X. Define the character $\theta \in X$ by $\theta(k) = k$, for $k \in K$. Define the transformation $T : X \to X$ to be the rotation by θ, i.e. $T(x) = \theta \cdot x$, where $(\theta \cdot x)(k) = \theta(k)x(k)$. In this way, $(X, \mathscr{B}, \mu, T)$ is a measure-preserving system. We need to show that such system has point-spectrum K.

First, we see that $K \subseteq spec(T)$. Let $k \in K$ be arbitrary. Pontryagin's Duality Theorem yields that K is isomorphic to $\widehat{X} = \widehat{\widehat{K}}$ given by $k \to f_k$ where $f_k(x) = x(k)$. Moreover, the characters for an orthonormal basis of $L^2(X, \mu)$. Therefore, for any character $f_k \in \widehat{X}$ and $x \in X$

$$(U_T f_k)(x) = f_k(Tx) = f_k(\theta \cdot x) = (\theta \cdot x)(k) = \theta(k) f_k(x) = k f_k(x).$$

Thus f_k is an eigenfunction of U_T with eigenvalue k, and hence $K \subseteq spec(T)$.

To prove the other inclusion, recall that for any compact abelian group G we have that $\widehat{G}$ forms an orthonormal basis for $L^2(G)$. This means that if f is an eigenfunction of U_T with eigenvalue $\lambda \in spec(T)$, then

$$f = \sum_{k\in K} c_k f_k,$$

where $c_k \in \mathbb{C}$ and the equality is in $L^2(X)$. Applying U_T and using the uniqueness of the coefficients, we have that

$$(\lambda - k)c_k = 0$$

for every $k \in K$. Thus $c_k = 0$ unless $k = \lambda$. Since $f \neq 0$, there is some $c_k \neq 0$, whence $\lambda = k$. Therefore $spec(T) \subseteq K$. □

P3. Let (X, A, μ, T) and (Y, B, ν, S) be two measure-preserving maps. Show that $T \times S$ has a discrete spectrum if and only if both T and S have a discrete spectrum.

If $T \times S$ has discrete spectrum, there is an orthonormal basis $\{h_n\}_{n\in\mathbb{N}}$ of eigenfunctions on $X \times Y$. Called π_X and π_Y the natural projections onto X and Y respectively. We consider the functions $y \to \{\int_X h_n(x,y)d\mu(x)\}_n$ and $x \to \{\int_Y h_n(x,y)d\nu(y)\}_n$ on the spaces Y and X respectively. We will construct an orthonormal basis of eigenfunctions on X and Y using those functions. We do this for X, since the same proof works for Y. Denote $f_n(x) = \int_Y h_n(x,y)d\nu(y)$.

- f_n is an eigenfunction: Let α_n the eigenvalue of h_n. Then,

$$f_n(Tx) = \int_Y h_n(Tx,y)d\nu(y) = \int_Y h_n(Tx,Sy)d\nu(y) = \alpha_n \int_Y h_n(x,y)d\nu(y)$$

  where we used the fact that S preserves ν in the middle equality.
- linear combinations of f_n are dense in $L^2(X)$: Let $f \in L^2(X)$ and consider the function $f \times 1 \in L^2(X \times Y)$. We know that for $\epsilon > 0$ there is a linear combination $\sum_{k=0}^K c_k h_k$ such that $||\sum_{k=0}^K c_k h_k - f \times 1||_{L^2(X\times Y)} \leq \epsilon$. In particular, using the Cauchy-Schwarz inequality we get

$$\begin{aligned}
\|\sum_{k=0}^K c_k f_k - f\|^2_{L^2(X)} &= \int_X \left| \int_Y \left(\sum_{k=0}^K c_k h_k(x,y) - f(x) \right) d\nu(y) \right|^2 d\mu(x) \\
&\leq \int_X \int_Y \left| \sum_{k=0}^K c_k h_k(x,y) - f(x) \right|^2 d\nu(y) d\mu(x) \\
&= \| \sum_{k=0}^K c_k h_k(x,y) - f \times 1\|^2_{L^2(X\times Y)} \leq \epsilon.
\end{aligned}$$

- Constructing an orthonormal subset: This collection of eigenfunctions is not necessarily orthogonal. However, as they are eigenfunctions, they are orthogonal if they correspond to different eigenvalues. So, we need to modify functions corresponding to the same eigenspace.

  For an eigenvalue $\alpha \in \mathcal{S}^1$, let $\{f_{n_k}\}_{k\in\mathbb{N}}$ the set of eigenfunctions associated to α (possibly a finite set). Define $\widetilde{f}_{n_1} = f_{n_1}$. Assume we have defined $\widetilde{f}_{n_j}$ for all $j < k$. Call $P_k = \sum_{j<k} \widetilde{f}_{n_j} \langle f_{n_k}, \widetilde{f}_{n_j} \rangle$ and define

$$\widetilde{f}_{n_k} = \begin{cases} (f_{n_k} - P_k)/||f_{n_k} - P_k|| & \text{if } f_{n_k} \neq P_k \\ 0 & \text{otherwise.} \end{cases}$$

  With this, after eliminating the zero functions, we have that $\{\tilde{f}_n\}_n$ is an orthonormal basis of eigenfunctions, and thus T has discrete spectrum.

Now suppose that T and S have discrete spectrum, with $\{f_n\}_n$ and $\{g_n\}_n$ orthonormal basis of eigenfunctions respectively. Consider $H = \{h_{n,m} := f_n \times g_m\}_{n,m\in\mathbb{N}}$. We claim that H is an orthonormal basis of eigenvalues of $X \times Y$. Indeed:

- Eigenfunction: Let α_n and β_m the eigenvalues of f_n and g_n. Then

$$(T \times S)h_n = Tf_n \cdot Sg_n = \alpha_n \beta_m f \cdot g = \alpha_n \beta_m h.$$

- Orthonormal: Let $(n_1, m_1), (n_2, m_2) \in \mathbb{N}^2$

$$\langle h_{n_1,m_1}, h_{n_2,m_2} \rangle = \int_X f_{n_1} f_{n_2} d\mu \cdot \int_Y g_{m_1} g_{m_2} d\nu = \begin{cases} 1 & \text{if } (n_1, m_1) = (n_2, m_2) \\ 0 & \text{otherwise} \end{cases}.$$

- Basis: As $\{f_n\}_n$ is dense in $L^2(X)$ and $\{g_m\}_m$ is dense in $L^2(Y)$ we have that the closure of H contains the set $\{f \times g \in L^2(X \times Y) \mid f \in L^2(X), g \in L^2(Y)\}$, which we know is dense in $L^2(X \times Y)$. Therefore $\overline{H} = L^2(X \times Y)$ and the proof is complete.

$\square$

P4. Let $(X, \mathscr{B}, \mu, T)$ and $(Y, \mathscr{A}, \nu, S)$ be measure-preserving systems. Show that $(X \times Y, \mathscr{B} \otimes \mathscr{A}, \mu \otimes \nu, T \times S)$ is weak mixing if and only if both $(X, \mathscr{B}, \mu, T)$ and $(Y, \mathscr{A}, \nu, S)$ are weak mixing.

($\Longleftarrow$) It is enough to prove that for $A, C \in \mathscr{B}$ and $B, D \in \mathscr{A}$ there is a subset $E \subseteq \mathbb{N}$ of zero upper density, such that

$$\lim_{\substack{n\to\infty \\ n\in E^c}} \mu \times \nu((T \times S)^{-n}(A \times B) \cap (C \times D)) = \mu \times \nu((A \times B)) \cdot \mu \times \nu(C \times D))$$

$$= \mu(T^{-n} A \cap C)\nu(S^{-n} B \cap D)$$

In fact, as X and Y are weakly mixing, there are sets E_1 and E_2 of zero upper density such that

$$\lim_{\substack{n\to\infty \\ n\in E_1^c}} \mu(T^{-n} A \cap C) = \mu(A)\mu(C), \text{ and } \lim_{\substack{n\to\infty \\ n\in E_2^c}} \nu(T^{-n} B \cap D) = \nu(B)\nu(D).$$

Consider $E = E_1 \cup E_2$. Notice that E has zero upper density. It follows that

$$\begin{aligned} \lim_{\substack{n\to\infty \\ n\in E^c}} \mu \times \nu((T \times S)^{-n}(A \times B) \cap (C \times D)) &= \lim_{\substack{n\to\infty \\ n\in E^c}} \mu(T^{-n} A \cap C)\nu(T^{-n} B \cap D) \\ &= \lim_{\substack{n\to\infty \\ n\in E^c}} \mu(T^{-n} A \cap C) \lim_{\substack{n\to\infty \\ n\in E^c}} \nu(T^{-n} B \cap D) \\ &= \mu \times \nu((A \times B)) \cdot \mu \times \nu(C \times D)) \end{aligned}$$

($\Longrightarrow$) For this direction, it is enough to see that any factor of a weak mixing system is weak mixing. For this, suppose that $(X, \mathscr{B}, \mu, T)$ is weak mixing and that $\pi\colon (X, \mathscr{B}, \mu, T) \to (Y, \mathscr{A}, \nu, S)$ is a factor. Let $A, B \in \mathscr{A}$, and consider $\pi^{-1}(A), \pi^{-1}(B) \in \mathscr{B}$. Using that X is weak mixing we have that there is a set of upper density zero E such that

$$\lim_{\substack{n\to\infty \\ n\in E^c}} \mu(T^{-n}\pi^{-1}(A) \cap \pi^{-1}(B)) = \mu(\pi^{-1}(A))\mu(\pi^{-1}(B)).$$

Using that $\pi\mu = \nu$ and $\pi T = S$ we get

$$\lim_{\substack{n\to\infty \\ n\in E^c}} \nu(S^{-n} A \cap B) = \nu(A)\nu(B),$$

and thus the factor is weak mixing as well.

P5. Let $(X, \mathscr{B}, \mu, T)$ be a measure-preserving system. Show that T^k is weak mixing if and only if T is weak mixing.

($\Longrightarrow$) If T is weak mixing then for $A, B \in \mathscr{B}$ we have

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}|\mu(T^{-kn}A\cap B)-\mu(A)\mu(B)| &\leq \frac{1}{N}\sum_{n=1}^{kN}|\mu(T^{-n}A\cap B)-\mu(A)\mu(B)| \\ &= \frac{kN}{N}\cdot\frac{1}{kN}\sum_{n=1}^{kN}|\mu(T^{-n}A\cap B)-\mu(A)\mu(B)| \\ &\to 0,\end{aligned}$$

as $N \to \infty$, concluding that T^k is weak mixing.

($\Longleftarrow$) If T^k is weak mixing then for $A, B \in \mathscr{B}$ and $i \in \{0, \ldots, k-1\}$ we have that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\mu(T^{-kn}T^{-i}A\cap B)-\mu(T^{-i}A)\mu(B)| = 0,$$

where the LHS be rewritten as

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\mu(T^{-(kn+i)}A\cap B)-\mu(A)\mu(B)|.$$

In particular, we can write $n = kl + i$ for $i \in \{0, \ldots, k-1\}$ and cleverly obtain

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\mu(T^{-n}A\cap B)-\mu(A)\mu(B)| \\ &= \lim_{N\to\infty}\sum_{i=0}^{k-1}\frac{1}{N}\sum_{l\leq\lfloor\frac{N-i}{k}\rfloor}|\mu(T^{-(kl+i)}A\cap B)-\mu(A)\mu(B)| \\ &= \sum_{i=0}^{k-1}\lim_{N\to\infty}\frac{\lfloor\frac{N-i}{k}\rfloor}{N}\cdot\frac{1}{\lfloor\frac{N-i}{k}\rfloor}\sum_{n\leq\lfloor\frac{N-i}{k}\rfloor}|\mu(T^{-(nk+i)}A\cap B)-\mu(A)\mu(B)| \\ &= 0,\end{aligned}$$

as T^k is weak-mixing.

P6. Let $(X, \mathscr{B}, \mu, T)$ be an invertible measure preserving system. Prove that if the system is weak mixing then for any set $A \in \mathscr{B}$ we have

$$\lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\mu(A\cap T^{-n}A\cap T^{-m}A\cap T^{-n-m}A) = \mu(A)^4.$$

Fix $A \in \mathcal{B}$ and fix $\epsilon > 0$. For each n let $B_n = A \cap T^{-n}A$. Let N_0 be such that, for $N > N_0$ we have

$$\frac{1}{N}\sum_{n=1}^{N}\left|\mu(B_n)-\mu(A)^2\right| < \frac{\epsilon}{4}$$

Note that for such N, using the $\triangle$-inequality, we get

$$\left|\frac{1}{N}\sum_{n=1}^{N}\mu(B_n)^2-\mu(A)^4\right| \le \frac{1}{N}\sum_{n=1}^{N}\left|\mu(B_n)^2-\mu(A)^4\right|$$
$$= \frac{1}{N}\sum_{n=1}^{N}\left|\mu(B_n)-\mu(A)^2\right|\left|\mu(B_n)+\mu(A)^2\right|$$
$$\le \frac{2}{N}\sum_{n=1}^{N}\left|\mu(B_n)-\mu(A)^2\right| < \frac{\epsilon}{2}.$$

For each N, let M_N be such that for $M > M_N$ we have, for each $n \le N$:

$$\frac{1}{M}\sum_{m=1}^{M}\left|\mu(B_n\cap T^{-m}B_n)-\mu(B_n)^2\right| < \frac{\epsilon}{2}.$$

Therefore, if $N > N_0$ and $M > M_N$, using the $\triangle$-inequality, we get

$$\left|\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\mu(A\cap T^{-n}A\cap T^{-m}A\cap T^{-n-m}A)-\mu(A)^4\right|$$
$$\le \left|\frac{1}{N}\sum_{n=1}^{N}\frac{1}{M}\sum_{m=1}^{M}\mu(B_n)^2-\mu(A)^4\right| + \left|\frac{1}{N}\sum_{n=1}^{N}\frac{1}{M}\sum_{m=1}^{M}\mu(B_n\cap T^{-m}B_n)-\mu(B_n)^2\right| < \epsilon.$$

P7. (a) Show that if the system $(X,\mathscr{B},\mu,T)$ is mixing, then for all strictly increasing sequences of integers n_k and any $A\in\mathscr{B}$ with $\mu(A)>0$, we have

$$\mu\left(\bigcup_{k=1}^{+\infty}T^{-n_k}A\right)=1.$$

Let $B=\bigcup_{k=1}^{+\infty}T^{-n_k}A$. Since the system is mixing, we have that

$$\lim_{n\to+\infty}\mu(T^{-n}A\cap B)=\mu(A)\mu(B).$$

In particular, we also have

$$\lim_{k\to+\infty}\mu(T^{-n_k}A\cap B)=\mu(A)\mu(B).$$

Let $\varepsilon>0$. If k is sufficiently large, we have that

$$\mu(T^{-n_k}A\cap B)\le\mu(A)\mu(B)+\varepsilon. \tag{5}$$

However, note that

$$T^{-n_k}A\cap B=\bigcup_{\ell=1}^{+\infty}(T^{-n_k}A\cap T^{-n_\ell}A)\supseteq T^{-n_k}A\cap T^{-n_k}A.$$

We infer that

$$\mu(T^{-n_k}A \cap B) \geq \mu(T^{-n_k}A) = \mu(A).$$

Combining this with (5), we get

$$\mu(A) \leq \mu(A)\mu(B) + \varepsilon \implies \mu(B) \geq \frac{\mu(A) - \varepsilon}{\mu(A)}.$$

Taking ε sufficiently small, we conclude that $\mu(B) = 1$.

(b) Show that if the system $(X, \mathscr{B}, \mu, T)$ is weak-mixing, then for all sequences of positive integers n_k with positive density and any $A \in \mathscr{B}$ with $\mu(A) > 0$, we have

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k}A\right) = 1.$$

(**Optional**) Show that the converse holds as well.

Let B be defined as above. Since the system is weak-mixing there exists a set $E \subseteq \mathbb{N}$, such that E^c has zero density and

$$\lim_{n \to +\infty, n \in E} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

We observe that E must contain infinitely many elements of the sequence n_k, as otherwise, the set E^c would contain all elements $n_K, n_{K+1}, ...$ for some $K \in \mathbb{N}$ and this would imply that E^c cannot have zero density.

Let n_{N_k} be a subsequence of n_k such that $n_{N_k} \in E$ for all $k \in \mathbb{N}$. We deduce that

$$\lim_{k \to +\infty} \mu(T^{-n_{N_k}}A \cap B) = \mu(A)\mu(B).$$

Let $\varepsilon > 0$. If k is sufficiently large, we have that

$$\mu(T^{-n_{N_k}}A \cap B) \leq \mu(A)\mu(B) + \varepsilon.$$

However, note that

$$T^{-n_{N_k}}A \cap B = \bigcup_{\ell=1}^{+\infty}(T^{-n_{N_k}}A \cap T^{-n_\ell}A) \supseteq T^{-n_{N_k}}A \cap T^{-n_{N_k}}A.$$

We infer that

$$\mu(T^{-n_{N_k}}A \cap B) \geq \mu(T^{-n_{N_k}}A) = \mu(A).$$

Similarly to part (a), we get

$$\mu(A) \leq \mu(A)\mu(B) + \varepsilon \implies \mu(B) \geq \frac{\mu(A) - \varepsilon}{\mu(A)}.$$

Taking ε sufficiently small, we conclude that $\mu(B) = 1$.

For the converse, we assume that for every set A of positive measure and any sequence n_k of positive density, we have $\mu(\cup_{k=1}^{+\infty} T^{-n_k}A) = 1$. We suppose that the system $(X, \mathscr{B}, \mu, T)$ is not weak-mixing and we will arrive at a contradiction.

For simplicity, let us say that a system $(X, \mathscr{B}, \mu, T)$ is "good" if for every set $A \in \mathscr{B}$ of positive measure and any sequence n_k of positive density, we have $\mu(\cup_{k=1}^{+\infty} T^{-n_k}A) = 1$.

The proof is a bit lengthy, but the strategy is as follows: if the system is not weak-mixing, it contains a factor that is isomorphic to a rotation on a compact abelian group. We show that the property of being good descends to this factor and then the problem reduces to showing that the rotation system does not possess the "good" property.

We prove the following claim:

Claim: A factor of a good system is also a good system.

Proof. To prove this claim, assume that we have a factor map $\pi : (X, \mathscr{B}, \mu, T) \to (Y, \mathscr{A}, \nu, S)$ where the first system is good. Let $B \in \mathscr{A}$ have positive measure and let n_k be any sequence with positive density. We want to show that

$$\nu\left(\bigcup_{k=1}^{+\infty} S^{-n_k}B\right) = 1.$$

We prove this by showing that

$$\nu\left(\bigcup_{k=1}^{N} S^{-n_k}B\right) = \mu\left(\bigcup_{k=1}^{N} T^{-n_k}A\right) \tag{6}$$

for all $N \in \mathbb{N}$, where $A = \pi^{-1}(B) \in \mathscr{B}$. Indeed, since π is a factor map, we have $\mu(A) = \mu(\pi^{-1}B) = \nu(B) > 0$. Then, taking the limit as $N \to +\infty$ and using the fact that the system $(X, \mathscr{B}, \mu, T)$ is good, we reach the desired conclusion.

In order to show (6), it suffices to prove that

$$\pi^{-1}\left(\bigcup_{k=1}^{N} S^{-n_k}B\right) = \bigcup_{k=1}^{N} T^{-n_k}A$$

and the claim would follow from the fact that ν is the pushforward of μ under π.

Let $x \in \bigcup_{k=1}^{N} T^{-n_k}A$ (x is a point on X). Then, we have that $T^{n_k}x \in A$ for some $k \in \{1, \dots, N\}$. Since π is a factor map, we have $\pi(T^{n_k}(x)) = S^{n_k}(\pi(x))$, so that $S^{n_k}(\pi(x)) \in \pi(A)$. As $A = \pi^{-1}(B)$, we deduce that $S^{n_k}(\pi(x)) = \pi(T^{n_k}x) \in B$. Therefore, $\pi(x) \in S^{-n_k}B \subseteq \bigcup_{k=1}^{N} S^{-n_k}B$, so that $x \in \pi^{-1}(\bigcup_{k=1}^{N} S^{-n_k}B)$.

Conversely, assume that $x \in \pi^{-1}\left(\bigcup_{k=1}^{N} S^{-n_k}B\right)$. Thus, $\pi(x) \in \bigcup_{k=1}^{N} S^{-n_k}B$ and we infer that there exists $k \in \{1, \dots, N\}$ such that $S^{n_k}\pi(x) \in B$. We deduce that $\pi(T^{n_k}x) \in B$ and thus, $T^{n_k}x \in \pi^{-1}(B) = A$. We conclude that $x \in \bigcup_{k=1}^{N} T^{-n_k}A$.

Combining the last two arguments, we deduce that

$$\pi^{-1}\left(\bigcup_{k=1}^{N} S^{-n_k}B\right) = \bigcup_{k=1}^{N} T^{-n_k}A$$

and the claim follows. □

We return to our exercise. Since the system is not weak-mixing, there exists an eigenfunction f with eigenvalue $e(a)$ for some $a \in [0, 1)$. Using exercise 4 a) from exercise sheet 7, we know that there exists a factor map $(X, \mathscr{B}, \mu, T) \to (Y, \mathscr{A}, \nu, S)$, such that the system $(Y, \mathscr{A}, \nu, S)$ is a rotation on a compact abelian group. In fact, an inspection of the argument in this exercise shows that we can take Y to the rotation by a on the torus $\mathbb{T}$, when a is irrational, or the rotation on q points, when $a = p/q, (p,q) = 1$ is rational. We conclude the exercise by showing that the last two types of systems are not good.

We start with the rotation on q points. Namely, we let $Y = \{0, \ldots, q-1\}$ with the uniform measure and the map $Sx = x + 1 \pmod q$. We let $A = \{0\}$ and take n_k to be the sequence of multiples of q. This has density $1/q$ and we also observe that $T^{-kq}A = 0$ for all $k \in N$, since $T^{kq}(0) = 0$. Therefore, we get that

$$\bigcup_{k=1}^{+\infty} T^{-kq}A = \{0\}$$

which has measure $1/q$. Thus, the rotation on q points is not good.

Now, let a be irrational and consider the rotation on the torus (with the Borel σ-algebra and the Lebesgue measure) by a. Namely, $Tx = x + a \pmod 1$. We let $A = [0, 1/4]$ and $B = [1/2, 3/4]$ (any pair of disjoint intervals works here, with slight modifications on the choice of R below). Consider the set

$$R = \left\{ n \in \mathbb{N} \colon \{na\} \in \left[0, \frac{1}{100}\right] \right\},$$

where $\{na\} = na - \lfloor na \rfloor$. Since a is irrational, the sequence na is uniformly distributed (mod 1), which implies that the set R has positive density (as it is equal to $\mu([0, 1/100]) = 1/100$). We pick the sequence n_k to be the elements of the set R in increasing order. For any $x \in A$, we have that

$$T^{n_k}x = x + n_k a \pmod 1 \in \left[0, \frac{1}{4} + \frac{1}{100}\right].$$

Therefore, we have that $T^{-n_k}A \cap B = \emptyset$ which implies that

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k}A\right) \leq \mu(\mathbb{T} \setminus B) \leq \frac{3}{4}.$$

We conclude that the rotation by a is not good.

To summarize, we have shown that if the original system is good but not weak-mixing, then it has a factor that is not good. This contradicts our claim, which implies that our original system is weak-mixing.

P8. (a) (**Optional**) Let $(X, \mathscr{B}, \mu, T)$ be a measure-preserving system. Suppose that for all $A, B \in \mathscr{B}$ there exists $n \in \mathbb{N}$ such that the following holds:

$$m \geq n \implies \mu(T^{-m}A \cap B) = \mu(A)\mu(B).$$

Prove that for all $A \in \mathscr{B}$, $\mu(A)$ is either 0 or 1.

[**Hint:** Use Baire's Category Theorem.]

Our goal is to use Baire's Category Theorem, so we state the version of the theorem that we will need.

Theorem 1. *Let X be a complete metric space. Then, every countable intersection of open dense sets is dense in X.*

We proceed by contradiction and assume that there exists $A \in \mathscr{B}$ such that $\mu(A) \in (0, 1)$.

Given $m \in \mathbb{N}$, let

$$\mathscr{B}_n = \{B \in \mathscr{B} \,|\, \mu(T^{-m}A \cap B) \neq \mu(A)\mu(B)\}.$$

We define the metric space (M, d) where $M = \mathscr{B}/\sim$ for the equivalence relation $A \sim B \iff \mu(A \Delta B) = 0$, and d is the metric given by $d([A],[B]) = \mu(A\Delta B)$. One can verify that this space is complete (**check!**).
It follows that for all $k \in \mathbb{N}$

$$\mathcal{H}_k = \bigcup_{m \geq k} \mathscr{B}_m \subseteq M$$

is open. One can verify that $\mathcal{H}_k$ is dense for k arbitrary. To see this, fix k, let $E \in \mathscr{B}$, and distinguish between two cases.

- $\underline{\boldsymbol{\mu(E) > 0}}$

  We proceed inductively. By assumption, there is $N \in \mathbb{N}$ such that

  $$\forall n \geq N \implies \mu(T^{-n}A \cap E) = \mu(A)\mu(E).$$

  Therefore, there is $n_1 \geq k$ such that $\mu(T^{-n_1}A \cap E) = \mu(A)\mu(E)$. We define $E_1 = T^{-n_1}A \cap E$. Now, suppose that there is $m \in \mathbb{N}$ and we are given $n_m \in \mathbb{N}$ and $E_m \subset E$ such that $\mu(E_m) = \mu(A)^m\mu(E)$. Therefore, by the same reasoning, there exists $n_{m+1} > n_m$ such that $\mu(E_{m+1}) = \mu(T^{-n_{m+1}}A \cap E) = \mu(A)\mu(E)$ where $E_{m+1} = T^{-n_{m+1}}A \cap E$.
  Observe that the sequence $(E_m)_{m\in\mathbb{N}}$ is a decreasing sequence of subsets of E and by construction the sequence $(n_m)_{m\in\mathbb{N}}$ is increasing. Moreover, by assumption $\mu(A)^m \to 0$ as $m \to \infty$. Therefore, $\forall \epsilon > 0$, $\exists M \in \mathbb{N}$ such that

  $$\forall m \geq M \quad \mu(E_m) = \mu(A)^m\mu(E) \in (0, \epsilon) \quad \text{and} \quad \mu(T^{-n_m}A \cap E) = \mu(A)\mu(E).$$

  Next, we observe that for every $m \geq M$ we have

  $$\begin{aligned}\mu(T^{-n_m}A \cap (E \setminus E_m)) = \mu(T^{-n_m}A \cap E) - \mu(E_m) &= \mu(A)\mu(E) - \mu(E_m) \\ &> \mu(A)\mu(E \setminus E_m).\end{aligned}$$

  Since $n_m > k$, we have that $E \setminus E_m \in \mathscr{B}_{n_m} \subseteq \mathcal{H}_k$. So $E \setminus E_m \in \mathcal{H}_k$ for every $m \geq M$. Moreover, for any $m \geq M$ we also have

  $$d([E],[E \setminus E_m]) = \mu(E\Delta(E \setminus E_m)) = \mu(E_m) < \epsilon.$$

- $\underline{\boldsymbol{\mu(E) = 0}}$

  Note that $\mathscr{B}_n$ is closed under taking complements (**check!**). Therefore, since we can approximate $X \setminus E$ by elements in $\mathcal{H}_k$ by the first case, we can also approximate E by elements in $\mathcal{H}_k$ as $\mathcal{H}_k$ is also closed under taking complements.

This concludes the proof that $\mathcal{H}_k$ is dense for k arbitrary.
By Baire's Category Theorem, $\mathcal{H} = \bigcap_{k\geq 1} \mathcal{H}_k = \limsup_k \mathscr{B}_k$ is a dense subset of M and in particular, it is non-empty. Therefore, for any $B \in \mathcal{H}$, there exist infinitely many n such that $\mu(T^{-n}A \cap B) \neq \mu(A)\mu(B)$, which yields a contradiction to our original assumption. Therefore, $\forall A \in \mathscr{B}$, $\mu(A) \in \{0, 1\}$.

(b) Suppose that for all $\varepsilon > 0$, there exists a natural number N_ε such that

$$\forall A, B \in \mathscr{B}, \forall n \geq N_\varepsilon \implies |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| < \varepsilon.$$

Again prove that for all $A \in \mathscr{B}$, $\mu(A)$ is either 0 or 1.

Let $A \in \mathscr{B}$ be arbitrary and choose $(n_k)_{k\in\mathbb{N}}$ increasing (assuming uniformity) such that

$$\forall B \in \mathscr{B} \quad |\mu(T^{-n_k}A \cap B) - \mu(A)\mu(B)| < \frac{1}{k}.$$

Let $k \in \mathbb{N}$ be arbitrary and set $B = T^{-n_k}A$. It follows that

$$|\mu(A) - \mu(A)^2| = |\mu(T^{-n_k}A) - \mu(A)^2| = |\mu(T^{-n_k}A \cap B) - \mu(A)\mu(B)| < \frac{1}{k},$$

which shows that $\mu(A) = \mu(A)^2$ as k was chosen arbitrarily, and thus the claim follows.