Ergodic Theory
Problem Sheet 8

Course Instructor: Florian K. Richter
Problems by: Jovan Andreevski

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For any questions or corrections, please send an email to: jovan.andreevski@epfl.ch!

P1. In this exercise, we identify the Pontryagin dual of $\mathbb{R}^n$.

- **(a)** We define for $a \in \mathbb{R}^n$ the map $\chi_a : x \in \mathbb{R}^n \mapsto e^{2\pi i x \cdot a}$. Prove that any character $\chi \in \widehat{\mathbb{R}^n}$ is of the form χ_a for some $a \in \mathbb{R}^n$.

  [**Hint:** Show that χ is smooth and find an ODE that is satisfied by χ.]

- **(b)** Prove that the map $a \in \mathbb{R}^n \mapsto \chi_a \in \widehat{\mathbb{R}^n}$ is a homeomorphism. This proves that the Pontryagin dual of $\mathbb{R}^n$ is $\mathbb{R}^n$.

P2. Show that given any countable subgroup $K \leq \mathbb{S}^1$, there exists a measure preserving system $(X, \mathcal{B}, \mu, T)$ on a Borel probability space such that K is the point-spectrum of T.

[**Hint:** To construct the system take $X = \widehat{K}$, $\mu = m_X$ the normalized Haar measure on X, and T to be some appropriate group rotation.]

P3. Let (X, A, μ, T) and (Y, B, ν, S) be two measure-preserving maps. Show that $T \times S$ has a discrete spectrum if and only if both T and S have discrete spectrum.

P4. Let $(X, \mathscr{B}, \mu, T)$ and $(Y, \mathscr{A}, \nu, S)$ be measure-preserving systems. Show that $(X \times Y, \mathscr{B} \otimes \mathscr{A}, \mu \otimes \nu, T \times S)$ is weak mixing if and only if both $(X, \mathscr{B}, \mu, T)$ and $(Y, \mathscr{A}, \nu, S)$ are weak mixing.

P5. Let $(X, \mathscr{B}, \mu, T)$ be a measure-preserving system. Show that T^k is weak mixing if and only if T is weak mixing.

P6. Let $(X, \mathscr{B}, \mu, T)$ be an invertible measure preserving system. Prove that if the system is weak mixing then for any set $A \in \mathscr{B}$ we have

$$\lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{NM} \sum_{n=1}^{N} \sum_{m=1}^{M} \mu(A \cap T^{-n}A \cap T^{-m}A \cap T^{-n-m}A) = \mu(A)^4.$$

P7. **(a)** Show that if the system $(X, \mathscr{B}, \mu, T)$ is mixing, then for all strictly increasing sequences of integers n_k and any $A \in \mathscr{B}$ with $\mu(A) > 0$, we have

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k}A\right) = 1.$$

(b) Show that if the system $(X, \mathscr{B}, \mu, T)$ is weak-mixing, then for all sequences of positive integers n_k with positive density and any $A \in \mathscr{B}$ with $\mu(A) > 0$, we have

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k} A\right) = 1.$$

(**Optional**) Show that the converse holds as well.

P8. **(a)** (**Optional**) Let $(X, \mathscr{B}, \mu, T)$ be a measure-preserving system. Suppose that for all $A, B \in \mathscr{B}$ there exists $n \in \mathbb{N}$ such that the following holds:

$$m \geq n \implies \mu(T^{-m} A \cap B) = \mu(A)\mu(B).$$

Prove that for all $A \in \mathscr{B}$, $\mu(A)$ is either 0 or 1.
[**Hint:** Use Baire's Category Theorem.]

(b) Suppose that for all $\varepsilon > 0$, there exists a natural number N_ε such that

$$\forall A, B \in \mathscr{B}, \forall n \geq N_\varepsilon \implies |\mu(T^{-n} A \cap B) - \mu(A)\mu(B)| < \varepsilon.$$

Again prove that for all $A \in \mathscr{B}$, $\mu(A)$ is either 0 or 1.