Ergodic Theory
Problem Sheet 7

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P1. Let $(X, \mathscr{B}, \mu, T)$ and $(Y, \mathscr{A}, \nu, S)$ be measure-preserving systems. Show that $(X \times Y, \mathscr{B} \otimes \mathscr{A}, \mu \otimes \nu, T \times S)$ is weak mixing if and only if both $(X, \mathscr{B}, \mu, T)$ and $(Y, \mathscr{A}, \nu, S)$ are weak mixing.

P2. Let $(X, \mathscr{B}, \mu, T)$ be a measure-preserving system. Show that T^k is weak mixing if and only if T is weak mixing.

P3. Let $(X, \mathscr{B}, \mu, T)$ be an invertible measure preserving system. Prove that if the system is weak mixing then for any set $A \in \mathscr{B}$ we have

$$\lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{NM} \sum_{n=1}^{N} \sum_{m=1}^{M} \mu(A \cap T^{-n}A \cap T^{-m}A \cap T^{-n-m}A) = \mu(A)^4.$$

P4. **(a)** Show that if the system $(X, \mathscr{B}, \mu, T)$ is mixing, then for all strictly increasing sequences of integers n_k and any $A \in \mathscr{B}$ with $\mu(A) > 0$, we have

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k}A\right) = 1.$$

(b) Show that if the system $(X, \mathscr{B}, \mu, T)$ is weak-mixing, then for all sequences of positive integers n_k with positive density and any $A \in \mathscr{B}$ with $\mu(A) > 0$, we have

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k}A\right) = 1.$$

(**Optional**) Show that the converse holds as well.

[**Hint:** For the converse, use the fact that if a system has an eigenfunction, then it has a factor map to a rotation system.]