Ergodic Theory
Solutions to Problem Sheet 6

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P1. a) Show that any factor of an ergodic system is ergodic. Find an example of a non-ergodic system with an ergodic factor.

> Let $(X, \mathscr{A}, \mu, T)$ be an ergodic measure preserving system, let $(Y, \mathscr{B}, \nu, S)$ be a factor, and let $\pi \colon X \to Y$ be the associated factor map. Let f be a S-invariant function. Then we have that
> $$f \circ \pi = f \circ S \circ \pi = f \circ \pi \circ T \quad \mu - \text{almost everywhere.}$$
> Since T is ergodic, it follows that $f \circ \pi$ is almost everywhere constant, and thus, f is ν - almost everywhere constant. By Proposition 45, the claim follows.
> Let $\alpha \in [0, 1)$ be irrational. The system $(\mathbb{T}^2, \mathscr{B}_{\mathbb{T}^2}, m_{\mathbb{T}^2}, R_\alpha \times R_\alpha)$ is not ergodic, but the rotation by α is an ergodic factor of this system (the associated map is the projection on the first coordinate). To see that this system is not ergodic, consider the function f given by $f(x, y) = x - y$, and notice that this is a non-constant invariant function.

b) Prove that the systems $\mathbf{X} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, R_\alpha)$ and $\mathbf{Y} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_2)$, where $R_\alpha x = x + \alpha \pmod 1$ and $T_2 x = 2x \pmod 1$, are not isomorphic.

> We proceed by contradiction. Suppose there exists a factor map $\phi \colon (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, R_\alpha) \to (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_2)$ that is invertible on a full measure subset X of $\mathbb{T}$. Consequently, $T_2 = \phi \circ R_\alpha \circ \phi^{-1}$ on X, which implies that T_2 is invertible on X. Take an arbitrary $x \in X$, then
> $$T_2\Big(x + \frac{1}{2}\Big) = 2x + 1 \pmod 1 = 2x \pmod 1 = T_2(x),$$
> and thus $x + \frac{1}{2} \notin X$. Therefore, using translation invariance of the Lebesgue measure, we obtain $m_{\mathbb{T}}(X) \leq \frac{1}{2}$, which is a contradiction.

c) Now, consider the systems $\mathbf{X} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_4)$ and $\mathbf{Y} = (\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), m_{\mathbb{T}^2}, T_2 \times T_2)$. Prove that they are isomorphic.

> We define the set of dyadic numbers $\mathbb{N}[\frac{1}{2}] = \{\frac{m}{2^n} : m, n \in \mathbb{N}; 0 \leq m < 2^n\}$ and let $X = [0, 1) \setminus \mathbb{N}[\frac{1}{2}]$. Notice that $m_{\mathbb{T}}(X) = 1$ and every element in X has a unique, non-terminating expansion in base 2. We define the map $\Phi : X \to X^2$ given by
> $$\Phi\left(\sum_{n=1}^{\infty} \frac{x_n}{2^n}\right) = \left(\sum_{n=1}^{\infty} \frac{x_{2n-1}}{2^n}, \sum_{n=1}^{\infty} \frac{x_{2n}}{2^n}\right),$$

for $x = 0.x_1x_2x_3\ldots$ in base 2. Then ϕ is a well-defined bijection between X and X^2, where the latter has full measure in $\mathbb{T}^2$.

First we prove that ϕ preserves the measure. For this, we identify (isomorphically) $\mathbb{T}$ with $\{0,1\}^{\mathbb{N}}$. Then we consider two cylinder sets A and B in $\{0,1\}^{\mathbb{N}}$, being the sets consisting of $0.x_1x_2x_3\ldots$ with the first k and m coordinates fixed, respectively. Hence, $\phi^{-1}(A\times B)$ has $k+m$ coordinates fixed (not necessarily the first $k+m$ consecutively). Therefore,

$$m_{\mathbb{T}}(\phi^{-1}(A\times B)) = 2^{-(k+m)} = 2^{-k}2^{-m} = m_{\mathbb{T}^2}(A\times B).$$

Finally, we show that $\phi\circ T_4 = (T_2\times T_2)\circ\phi$. For $x = 0.x_1x_2x_3\ldots \in X$, we have

$$\begin{aligned}\phi\circ T_4(x) = \phi(x_1x_2.x_3x_4\ldots \pmod 1) &= \phi(0.x_3x_4x_5\ldots)\\ &= (0.x_3x_5\ldots, 0.x_4x_6\ldots).\end{aligned}$$

On the other hand,

$$\begin{aligned}(T_2\times T_2)\circ\Phi(x) = (T_2(0.x_1x_3\ldots), T_2(0.x_2x_4\ldots)) &= (x_1.x_3x_5 \pmod 1, x_2.x_4x_6\ldots \pmod 1)\\ &= (0.x_3x_5\ldots, 0.x_4x_6\ldots).\end{aligned}$$

This completes the proof.

P2. Let $\{\alpha_\ell\}_{\ell\in\mathbb{N}} \subseteq \mathbb{T}$. Show that there is an increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that for every $k\in\mathbb{N}$,

$$\|n_k\alpha_\ell\|_{\mathbb{T}} \leq \frac{1}{k}, \quad \forall \ell\in\{1,\ldots,k\}.$$

where for $t\in\mathbb{T}$ we denote $\|t\|_{\mathbb{T}} = \text{mín}(1-t, t)$.

[**Hint:** Use Poincaré's Recurrence Theorem in a convenient group rotation.]

Define $n_1\in\mathbb{N}$ arbitrarily. Assume that we have defined $n_{k-1}\in\mathbb{N}$ holding the aforementioned property. Consider the group rotation $(\mathbb{T}^k, \mathscr{B}(\mathbb{T}^k), \mu, R)$, where μ is the Haar measure on $\mathbb{T}^k$ (in this case, the projection of the Lebesgue measure from $\mathbb{R}^k$) and R is the rotation by the element $(\alpha_1,\ldots,\alpha_k)$. The distance in $\mathbb{T}^k$ given by $d(x,y) = \text{máx}_{i\in[k]}\|x_i - y_i\|_{\mathbb{T}}$ generates the topology of $\mathbb{T}^k$.

Set $U = B(0, \frac{1}{2k})$, which has positive measure. Then by Poincaré's Reccurence Theorem, there exists $n_k > n_{k-1}$ such that

$$\mu(U\cap R^{-n_k}U) > 0.$$

This implies that there exists $x\in U$ such that $R^{n_k}x\in U$. Therefore, using the triangle inequality, we obtain that

$$\|(n_k\alpha_1,\ldots,n_k\alpha_k)\|_{\mathbb{T}^k} = \|R^{n_k}x - x\|_{\mathbb{T}^k} < \frac{1}{k},$$

which implies

$$\|n_k\alpha_\ell\|_{\mathbb{T}} \leq \frac{1}{k}, \quad \forall\ell\in\{1,\ldots,k\}.$$

P3. In this exercise, we consider two explicit examples of a Haar measure. Additionally, briefly argue why the provided groups satisfy the properties required by Haar's Theorem.

a) Consider the group $G = \mathrm{SL}_n(\mathbb{R})$, which we can view as a closed subset of $\mathrm{Mat}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

Let λ denote the Lebesgue measure on $\mathrm{Mat}_{n\times n}(\mathbb{R})$. For a measurable $A \subset G$, we define

$$m_G(A) = \lambda(\{tg \mid t \in [0,1],\, g \in A\}).$$

Prove that m_G is a left Haar measure on G.
Is m_G also right-invariant? If yes, provide a proof. If not, provide an example of a right Haar measure.

First, we define the map

$$\phi : \mathbb{R}_{>0} \times \mathrm{SL}_n(\mathbb{R}) \mapsto \mathrm{GL}_n^+(\mathbb{R})$$
$$(k, g) \mapsto kg$$

and notice that ϕ is a homeomorphism with inverse

$$g \in \mathrm{GL}_n(\mathbb{R}) \mapsto (|\det(g)^{\frac{1}{n}}|, |\det(g)^{\frac{1}{n}}|^{-1} \cdot g).$$

Furthermore, notice that $\mathrm{GL}_n^+(\mathbb{R})$ is an open subset of $\mathrm{Mat}_{n\times n}(\mathbb{R})$, and thus any non-empty subset of $\mathrm{GL}_n^+(\mathbb{R})$ has positive Lebesgue measure.
Now, we check the properties of a Haar measure.

i) Let $K \subset G$ be compact. Then, by definition $m_G(K) = \lambda([0,1]\cdot K)$. Notice that $[0,1]\cdot K$ is compact as it is the image of $[0,1]\times K$ under ϕ. In particular, it has finite Lebesgue measure, and consequently, so does K.

ii) Let $U \subset G$ be open. Using the same line of reasoning as before, we note that $(0,1)\cdot U$ is a non-empty open subset of $\mathrm{GL}_n(\mathbb{R})$, and thus has positive Lebesgue measure. By definition, we obtain

$$m_G(U) = \lambda([0,1]\cdot U) = \lambda((0,1)\cdot U) > 0.$$

iii) Recall that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map and $E \subset \mathbb{R}^n$ is measurable, then $\lambda(T(E)) = |\det T|\lambda(E)$. Now, observe that λ is $\mathrm{SL}_n(\mathbb{R})$-invariant in the following sense.
Using our identification $\mathrm{Mat}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$, let $g \in \mathrm{SL}_n(\mathbb{R})$ and define the linear map $L_g : \mathrm{Mat}_{n\times n}(\mathbb{R}) \mapsto \mathrm{Mat}_{n\times n}(\mathbb{R})$, $L_g(h) = gh$. In coordinates and using vectorization, we have that $\mathrm{vec}(gh) = (I_n \bigotimes g)\mathrm{vec}(h)$, where I_n is the $n\times n$ identity matrix, and $I_n \bigotimes g$ is the Kronecker product of I_n and g. Therefore, L_g is the linear map given by the matrix $I_n \bigotimes g$, and its determinant is precisely $\det(I_n)^n \det(g)^n = 1$, since $g \in \mathrm{SL}_n(\mathbb{R})$. Therefore, $\lambda(L_g(E)) = \lambda(E)$ for any measurable $E \subset \mathbb{R}^n$. The same invariance property also holds for the map $R_g(h) = hg$, where $g \in \mathrm{SL}_n(\mathbb{R}), h \in \mathrm{Mat}_{n\times n}(\mathbb{R})$.

Now, let $g \in \mathrm{SL}_n(\mathbb{R})$ and $B \subset \mathrm{SL}_n(\mathbb{R})$ be measurable. We notice that $[0,1]\cdot B \subset \mathrm{Mat}_{n\times n}(\mathbb{R})$ is measurable and obtain

$$m_G(gB) = \lambda([0,1]\cdot(gB)) = \lambda(g([0,1]\cdot B)) = \lambda([0,1]\cdot B) = m_G(B),$$

thus showing that m_G is left translation invariant. One obtains that m_G is right translation invariant in an analogous manner, which means that $G = \mathrm{SL}_n(\mathbb{R})$ is unimodular.

b) Consider the group $H = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{R} \setminus \{0\},\, b \in \mathbb{R} \right\}$. Recalling that we can define measures in terms of how they integrate integrable functions, show that the measure m_H defined by $\mathrm{d}m_H = \frac{1}{a^2}\mathrm{d}a\,\mathrm{d}b$ defines a left Haar measure on H.

Is m_H also right-invariant? If yes, provide a proof. If not, provide an example of a right Haar measure.

> First, we observe that H is homeomorphic to $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$.
>
> Out of the three defining properties of a Haar measure, proving left translation invariance is the most interesting. The other two properties easily follow, so we leave out the details.
>
> Let $f \in L^1_{m_H}(H)$ and $h \in H$ with $h = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ for some $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$. Then,
>
> $$\begin{aligned} \int_H f(hg)\mathrm{d}m_H(g) &= \int_H f\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \right) \frac{1}{x^2}\mathrm{d}x\mathrm{d}y \\ &= \int_H f\left(\begin{bmatrix} ax & ay+b \\ 0 & 1 \end{bmatrix} \right) \frac{1}{x^2}\mathrm{d}x\mathrm{d}y \\ &= \int_H f\left(\begin{bmatrix} s & t \\ 0 & 1 \end{bmatrix} \right) \frac{a^2}{s^2}\frac{1}{a}\mathrm{d}s\frac{1}{a}\mathrm{d}t \\ &= \int_H f\left(\begin{bmatrix} s & t \\ 0 & 1 \end{bmatrix} \right) \frac{1}{s^2}\mathrm{d}s\mathrm{d}t \\ &= \int_H f(g)\mathrm{d}m_H(g), \end{aligned}$$
>
> where we made the substitution $s = ax, t = ay + b$. This shows that our measure m_H is left-translation invariant. In similar fashion, one can show that $\mathrm{d}m_H' = \frac{1}{a}\mathrm{d}a\mathrm{d}b$ is a right Haar measure on H. This measure is not proportional to m_H, and therefore m_H is not a right Haar measure on H.

P4. Let $(X, \mathscr{B}, \mu, T)$ be an ergodic measure preserving system, and let $\alpha \in (0, 1)$ such that $e(\alpha)$ is an eigenvalue. Show that there exists a non-trivial group rotation that is a factor of $(X, \mathscr{B}, \mu, T)$.

[**Hint:** When $\alpha = r/q$ is rational (with q minimal among all such rational eigenvalues), construct a T^q-invariant set B such that $\mu(B) = 1/q$.]

> Let $f \in L^2(X)$ be the eigenfunction corresponding to the given eigenvalue. Then $Tf = e(\alpha)f$ and f is non-constant.
>
> Suppose first that $\alpha \in \mathbb{Q}$. We write $\alpha = r/q$ for some $0 < r < q$ with $(r, q) = 1$. Then $T^q f = f$ and thus the transformation T^q is not ergodic. Let $q \geq 2$ be the minimal integer (larger than 1) for which T^q is not ergodic and consider $A \in \mathscr{B}$ such that $T^{-q}A = A$ and $\mu(A) \in (0, 1)$.
>
> Define $g = \sum_{n=0}^{q-1} T^n \mathbb{1}_A$ and notice that g is T-invariant. It follows by ergodicity that g is almost everywhere equal to some integer constant, and then $q\mu(A) = \int g \, d\mu \in \mathbb{Z}$. Thus, $\mu(A) = k/q$ for some $k \in \{1, \dots, q-1\}$.
>
> <u>Claim</u>. There exists a T^q-invariant set $B \in \mathscr{B}$ such that $\mu(B) = 1/q$.
>
> To prove the claim, suppose that $k > 1$, since otherwise the claim holds trivially. First, we notice that by the pigeonhole principle, there exists some $m \in \{1, \dots, q-1\}$ such that $\mu(A_1) > 0$, where $A_1 = A \cap T^{-m}A$. If $\mu(A_1) = \mu(A)$, then $T^{-m}A = A$ almost everywhere, which contradicts the minimality of q. Then $0 < \mu(A_1) < \mu(A) = k/q$. We can define g_1 as

we defined g substituting A with A_1, and since A_1 is also T^q-invariant, then as before, there exists some integer $0 < k_1 < k$ such that $\mu(A_1) = k_1/q$. If $k_1 = 1$, then the claim follows by taking $B = A_1$. Otherwise we repeat the same argument for A_1 instead of A, to find some set A_2 with $\mu(A_2) = k_2/q$ for some $0 < k_2 < k_1$. Inductively, we find $j \in \{1, \dots, q-1\}$ such that $B = A_j$ satisfies the claim.

Now, consider the set $Y = \{0, \dots, q-1\}$, let $\mathscr{B}_Y$ be the discrete σ-algebra on Y, ν be the normalized counting measure on Y, and let $S : y \mapsto y + 1 \pmod{q}$. Then $(Y, \mathscr{B}_Y, \nu, S)$ is a measure-preserving system (a rotation on finitely many points). Using our claim, we deduce that the sets $B, T^{-1}B, \dots, T^{q-1}B$ form a partition of X, thus for any $x \in X$, there exists a unique $y_x \in Y$ such that $x \in T^{-y_x}B$. We can then define the map $\pi : X \to Y$ by $\pi(x) = y_x$, and it is not hard to check that this is a factor map.

Now suppose that $\alpha \notin \mathbb{Q}$. We show that $(\mathbb{T}, \mathscr{B}_{\mathbb{T}}, m_{\mathbb{T}}, R_\alpha)$ is a factor of $(X, \mathscr{B}, \mu, T)$. We identify $\mathbb{T}$ with $[0, 1)$ for simplicity. Let $\pi : X \to \mathbb{T}$ be the map defined uniquely by the equality $f(x) = e(\pi(x))$. Observe that this is well-defined for almost all $x \in X$, since $|f(x)| = 1$ almost everywhere.

We observe that

$$e(\pi(Tx)) = f(Tx) = e(\alpha)f(x) = e(\pi(x) + \alpha) = e(R_\alpha \pi(x)),$$

which means that $\pi(Tx) = R_a\pi(x)$ almost everywhere. Finally, π preserves $m_{\mathbb{T}}$. To see this, define a measure ν on $\mathbb{T}$ by $\nu(A) = \mu(\pi^{-1}A)$, $A \in \mathscr{B}$. We easily check that ν is R_α-invariant, and so it is the Haar measure $m_{\mathbb{T}}$ (the only measure invariant under an irrational rotation is the Haar measure).