Ergodic Theory
Problem Sheet 6

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P1. a) Show that any factor of an ergodic system is ergodic. Find an example of a non-ergodic system with an ergodic factor.

b) Prove that the systems $\mathbf{X} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, R_\alpha)$ and $\mathbf{Y} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_2)$, where $R_\alpha x = x + \alpha \pmod 1$ and $T_2 x = 2x \pmod 1$, are not isomorphic.

c) Now, consider systems $\mathbf{X} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_4)$ and $\mathbf{Y} = (\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), m_{\mathbb{T}^2}, T_2 \times T_2)$. Prove that they are isomorphic.

P2. Let $\{\alpha_\ell\}_{\ell \in \mathbb{N}} \subseteq \mathbb{T}$. Show that there is an increasing sequence $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that for every $k \in \mathbb{N}$,

$$\|n_k \alpha_\ell\|_{\mathbb{T}} \leq \frac{1}{k}, \quad \forall \ell \in \{1, \dots, k\},$$

where for $t \in \mathbb{T}$ we denote $\|t\|_{\mathbb{T}} = \min(1 - t, t)$.

[**Hint:** Use Poincaré's Recurrence Theorem in a convenient group rotation.]

P3. In this exercise, we consider two explicit examples of a Haar measure. Additionally, briefly argue why the provided groups satisfy the properties required by Haar's Theorem.

a) Consider the group $G = \mathrm{SL}_n(\mathbb{R})$, which we can view as a closed subset of $\mathrm{Mat}_{n \times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Let λ denote the Lebesgue measure on $\mathrm{Mat}_{n \times n}(\mathbb{R})$. For a measurable $A \subset G$, we define

$$m_G(A) = \lambda(\{tg \mid t \in [0, 1],\, g \in A\}).$$

Prove that m_G is a left Haar measure on G.

Is m_G also right-invariant? If yes, provide a proof. If not, provide an example of a right Haar measure.

b) Consider the group $H = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{R} \setminus \{0\},\, b \in \mathbb{R} \right\}$. Recalling that we can define measures in terms of how they integrate integrable functions, show that the measure m_H defined by $\mathrm{d}m_H = \frac{1}{a^2} \mathrm{d}a\, \mathrm{d}b$ defines a left Haar measure on H.

Is m_H also right-invariant? If yes, provide a proof. If not, provide an example of a right Haar measure.

P4. Let $(X, \mathscr{B}, \mu, T)$ be an ergodic measure preserving system, and let $\alpha \in (0, 1)$ such that $e(\alpha)$ is an eigenvalue. Show that there exists a non-trivial group rotation that is a factor of $(X, \mathscr{B}, \mu, T)$.

[**Hint:** When $\alpha = r/q$ is rational (with q minimal among all such rational eigenvalues), construct a T^q-invariant set B such that $\mu(B) = 1/q$.]