

# Ergodic Theory

## Problem Sheet 4

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**P1.** In this exercise, we prove a version of the point ergodic theorem for non-integrable functions.

Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic measure-preserving system. Suppose  $f \geq 0$  is a measurable function such that  $\int f \, d\mu = +\infty$  and define

$$f^*(x) = \liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

as well as the set

$$A = \{x \in X : f^*(x) < +\infty\}$$

(a) Show that  $f^*$  is  $T$ -invariant.

[**Hint:** Start with the identity

$$\frac{N+1}{N} \left( \frac{1}{N+1} \sum_{n=0}^N f(T^n x) \right) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(Tx)) + \frac{1}{N} f(x),$$

for  $N \in \mathbb{N}$ ,  $x \in X$ .]

(b) Show that the function  $f^* \cdot \mathbb{1}_A$  is constant almost everywhere on  $X$ .

(c) Conclude that  $\mu(A) = 0$  and thus

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = +\infty$$

for almost all  $x \in X$ .

[**Hint:** Construct an increasing sequence of bounded functions  $f_m$  that converges to  $f$  pointwise.]

**P2.** Consider the system  $(\mathbb{T}, \mathcal{B}_{\mathbb{T}}, \mu, T)$  where  $\mu$  is the Lebesgue measure and  $T(x) = qx \bmod 1$ , for  $q \geq 2$  a fixed integer. If  $x \in \mathbb{R}$  we say that  $x$  is written in its  $q$ -ary digit expansion if

$$x = [x] + \sum_{j=1}^{\infty} a_j(x) q^{-j},$$

for  $\{a_j\}_{j \in \mathbb{N}} \subseteq \{0, 1, \dots, q-1\}$ , such that for all  $J \in \mathbb{N}$  there exists  $j \geq J$  such that  $a_j \neq q-1$ .

(a) For a given  $x \in \mathbb{R}$ , prove that  $a_j(x) = i$  if and only if  $T^{j-1}x \in [\frac{i}{q}, \frac{i+1}{q})$  with  $i \in \{0, 1, \dots, q-1\}$ . Additionally, prove that  $a_j(Tx) = a_{j+1}(x)$  for all  $x \in \mathbb{R}$  and  $j \geq 1$ .

(b) **Optional question:** We call a real number  $x$  simply normal in base  $q$  if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{j \leq N : a_j(x) = d\} = \frac{1}{q},$$

for all  $d = 0, \dots, q - 1$  (i.e. all digits  $d$  appear asymptotically with equal frequencies  $1/q$ ). A number  $x$  is called  $q$ -normal if for every  $i, j \in \mathbb{N}$ ,  $q^i x$  is simply normal with respect to  $q^j$ . Show that if  $x$  is  $q$ -normal, then  $\{q^n x\}_n$  is uniformly distributed with respect to  $\mu$ .

**P3.** Consider the transformation  $T: \mathbb{T}^2 \rightarrow \mathbb{T}^2$  given by  $T(x, y) = (x + \alpha, y + 2x + \alpha) \bmod 1$ .

- (a) Show that  $T^n(x, y) = (x + n\alpha, y + 2nx + n^2\alpha) \bmod 1$ .
- (b) Show that  $T$  preserves the Lebesgue measure  $\mu \times \mu$  on  $\mathbb{T}^2$ .
- (c) From now on, assume  $\alpha \notin \mathbb{Q}$ . Show that  $(\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \mu \times \mu, T)$  is ergodic.
- (d) Show that if  $f \in L^1(\mathbb{T})$  then for  $\mu \times \mu$ -almost every  $(b, c) \in \mathbb{T}^2$  we have

$$\frac{1}{N} \sum_{n=0}^N f(\alpha n^2 + bn + c) \xrightarrow{N \rightarrow \infty} \int_{\mathbb{T}} f d\mu.$$

- (e) Conclude that for  $\mu \times \mu$ -almost every  $(b, c) \in \mathbb{T}^2$ , the sequence  $\{\alpha n^2 + bn + c\}_{n \in \mathbb{N}}$  is equidistributed mod 1.

**P4.** Let  $(X, \mathcal{B}, \mu, T)$  be a measure preserving system,  $f \in L^1(\mu)$  and  $a \in \mathbb{R}$ . Show that, for almost all  $x \in X$ , the limit

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(na) f(T^n x)$$

exists.