Ergodic Theory
Problem Sheet 3

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P1. Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving system and let $A \in \mathcal{A}$ be a set of positive measure. Prove Khintchine's theorem: for any $\varepsilon > 0$, the set

$$\{n \in \mathbb{N} \colon \mu(A \cap T^{-n}A) \geq (\mu(A))^2 - \varepsilon\}$$

has bounded gaps.

P2. Prove that the sequence $(x_n)_{n\in\mathbb{N}}$ is uniformly distributed mod 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\{x_n\}^h = \frac{1}{h+1}, \ \forall h \in \mathbb{N}.$$

P3. **(a)** Prove that the sequence $(\log n)_{n\in\mathbb{N}}$ is not uniformly distributed mod 1.

The rest of this problem is **optional**.

(b) We say that a sequence $x_n \in [0, 1)$ is uniformly distributed with respect to logarithmic averages[1] if for every $0 \leq a \leq b \leq 1$, we have

$$\lim_{N\to\infty} \frac{1}{\log N}\sum_{n=1}^{N}\frac{\mathbb{1}_{[a,b)}(x_n)}{n} = b - a.$$

Prove that if a sequence is uniformly distributed in the classical sense, then it is uniformly distributed with respect to logarithmic averages.

[**Hint:** Use summation by parts.]

(c) Prove that the sequence $\log n \pmod 1$ is uniformly distributed with respect to logarithmic averages. Conclude, in particular, that the sequence $\{\log n\}$ is dense in $[0, 1]$ (you may assume without proof in this exercise that Weyl's criterion holds for logarithmic averages).

P4. Let (y_n) be a sequence of distinct integers. For every $m \in \mathbb{N}$ define the sequence of functions

$$S_{m,N}(x) = \frac{1}{N}\sum_{n=1}^{N} e(my_n x).$$

(a) Prove that $S_{m,N}(x) \to 0$ as $N \to +\infty$ for almost all $x \in [0, 1]$.

[**Hint:** Compute $\|S_{m,N^2}(x)\|_{L^2([0,1))}$ and show that $S_{m,N^2}(x) \to 0$ for almost all (with respect to the Lebesgue measure) $x \in [0, 1]$.]

(b) Prove that for almost all $a \in [0, 1]$, the sequence $(y_n a)_{n\in\mathbb{N}}$ is uniformly distributed mod 1. As an application, conclude that for almost all real numbers the sequence $\{b^n x\}$ is uniformly distributed mod 1 for all $b \in \mathbb{N}$ with $b \geq 2$.

[1] Compare this to the classical notion of uniform distribution using the Cesàro averages $\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{[a,b)}(x_n)$.