Ergodic Theory
Problem Sheet 2

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P1. Let $(X, \mathscr{A}, \mu, T)$ be a measure-preserving system.

(a) Let $A, B \in \mathscr{A}$. Show that if $(X, \mathscr{A}, \mu, T)$ is ergodic then

$$\mu(A) - \sqrt{\mu(A)(1-\mu(B))} \leq \limsup_{n\to\infty} \mu(T^{-n}A \cap B) \quad \text{and} \quad \liminf_{n\to\infty} \mu(T^{-n}A \cap B) \leq \mu(A)\mu(B).$$

(b) Let $f \in L^2(X, \mathscr{A}, \mu)$. Show that the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int U_T^n f \cdot \overline{f} \, d\mu$$

exists, is real, and is greater or equal than $|\int f \, d\mu|^2$.

P2. **(a)** Show that the circle rotation system $(\mathbb{T}, \mathscr{B}_{\mathbb{T}}, m, R_\alpha)$ is ergodic if and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

(b) Show that the circle doubling system $(\mathbb{T}, \mathscr{B}_{\mathbb{T}}, m, T)$, where $T_2(x) = 2x \mod 1$, is ergodic. [**Hint:** Use Fourier expansion.]

P3. Let $(X, \mathscr{A}, \mu, T)$ be a measure-preserving system. Show that for any $f \in L^1(X)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U_T^n f = f^* \quad \text{in } L^1(X), \tag{1}$$

for some T-invariant function $f^* \in L^1(X)$.

P4. Let $(X, \mathscr{A}, \mu, T)$ be a measure-preserving system. We call $(X, \mathscr{A}, \mu, T)$ **mixing** if for all $A, B \in \mathscr{A}$, $\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B)$. Show that $(X, \mathscr{A}, \mu, T)$ is mixing if and only if for all $A \in \mathscr{A}$ it holds

$$\lim_{n\to\infty} \mu(T^{-n}A \cap A) = \mu(A)^2. \tag{2}$$

[**Hint:** Observe that it is enough to show

$$\lim_{n\to\infty} \int_X U_T^n \mathbb{1}_A \cdot f \, d\mu = \mu(A) \cdot \int f \, d\mu$$

for all $f \in L^2(\mu)$, and that it suffices to establish this for all $f \in \overline{\langle \{c \colon c \in \mathbb{R}\} \cup \{U_T^k \mathbb{1}_A \colon k \in \mathbb{N}_0\} \rangle}$.]

P5. Let X be a compact metric space, and let $T : X \to X$ be continuous. Suppose that μ is a T-invariant ergodic probability measure defined in the Borel subsets of X. Prove the following:

(a) The support of the measure μ, defined as

$$\operatorname{supp}(\mu) = X \setminus \left(\bigcup_{\substack{U \subseteq X \text{ open} \\ \mu(U)=0}} U \right),$$

has full measure.

(b) For μ-almost every $x \in X$ and for every $y \in \operatorname{supp}(\mu)$, there exists a sequence $n_k \nearrow \infty$ such that $T^{n_k}x \to y$ as $k \to \infty$.