Ergodic Theory
Problem Sheet 1

September 8, 2025

P1. Let $(X, \mathcal{A}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable map. Show that T preserves μ if and only if

$$\int f \circ T d\mu = \int f d\mu \tag{1}$$

holds for any $f \in L^1(X)$.

Optional question: Show that if X is a Polish space (Hausdorff completely metrizable topological space) and μ is a Borel probability measure on X, then show that it suffices to check (1) for any $f \in C(X)$. [**Hint:** Use the fact that any finite Borel measure on a Polish space is regular, namely for any measurable set A and any $\varepsilon > 0$ there exist open U and compact K such that $K \subset A \subset U$ and $\mu(U \setminus K) < \varepsilon$.]

P2. (a) Let $(X, \mathcal{A}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable map. Assume that $\mathcal{S}$ is a semi-algebra that generates $\mathcal{A}$. Show that if $\mu(T^{-1}A) = \mu(A)$ for any $A \in \mathcal{S}$, then T preserves μ.

(b) Deduce that the product system of two measure-preserving systems is also measure-preserving.

P3. We consider the torus with the Borel σ-algebra and the Lebesgue measure.

(a) Show that for any $\alpha \in \mathbb{R}$, the map $Tx = x + \alpha \pmod 1$ preserves the Lebesgue measure.

(b) For each $p \in \mathbb{N}$, we define the map $T_p x = px \pmod 1$ for all $x \in [0, 1)$. Show that the transformation T_p preserves the Lebesgue measure.

P4. Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving system, with T invertible. Now, at any moment, instead of moving forward by T, we flip a fair coin to decide whether we will use T or T^{-1}.

The goal is to describe the random system described above by means of a measure-preserving system. In particular, we want to find a map R such that given a point x and a sequence of coin tosses ω, we would have that $R(\omega, x)$ would produce the same result as the procedure above.

(a) Find a measure-preserving system $(Y, \mathcal{B}, \nu, S)$ that models the sequence of coin tosses.

(b) Consider the product system $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$. Define a measure-preserving map R on this product space that models the original random system.

P5. A set $R \subseteq \mathbb{Z}$ is a set of recurrence if for every measure-preserving system $(X, \mathcal{A}, \mu, T)$ and for all $A \in \mathcal{A}$ with $\mu(A) > 0$ we have $\mu(A \cap T^{-n}A) > 0$ for some $n \in R \setminus \{0\}$.

(a) Show that $2\mathbb{N}$ is a set of recurrence, but $2\mathbb{N} + 1$ is not.

(b) Show that if $E \subseteq \mathbb{N}$ is an infinite set, then $E - E = \{a - b \colon a, b \in E\}$ is a set of recurrence.

(c) Show that sets of recurrence possess the Ramsey property: If $R = R_1 \cup R_2 \cup \cdots \cup R_k$ is a set of recurrence, then one of the sets $R_1, R_2, \ldots, R_k$ is a set of recurrence.

P6. Prove the following strengthening of Poincaré's Recurrence Theorem: For any measure-preserving system $(X, \mathcal{A}, \mu, T)$, any $A \in \mathcal{A}$, and any $\varepsilon > 0$, there exists some $n \in \mathbb{N}$ such that

$$\mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon.$$