Ergodic Theory
Solutions to Problem Sheet 13

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P1. Let $(X, \mathscr{B}, \mu)$ be a probability space, $\mathscr{A} \subseteq \mathscr{B}$ be a σ-algebra and $f \in L^1(X, \mathscr{B}, \mu)$ be a real-valued function. Let $m = \inf_{y \in X} f(y)$ and $M = \sup_{y \in X} f(y)$. Show that $\mathbb{E}(f \mid \mathscr{A})(x) \in [m, M]$ for almost every $x \in X$.

> Notice that it is enough to prove just the lower bound as the upper bound follows from taking $-f$. For $\epsilon > 0$, define
>
> $$A_\epsilon = \{x \in X \mid \mathbb{E}(f \mid \mathscr{A})(x) < m - \epsilon\} \in \mathscr{A}.$$
>
> Then, notice that
>
> $$m\mu(A_\epsilon) \leq \int_{A_\epsilon} f d\mu = \int_{A_\epsilon} \mathbb{E}(f \mid \mathscr{A}) d\mu \leq (m - \epsilon)\mu(A_\epsilon),$$
>
> which implies $\mu(A_\epsilon) = 0$. Then we notice that
>
> $$\bigcap_{k \in \mathbb{N}} A_{1/k} = \{x \in X \mid \mathbb{E}(f \mid \mathscr{A})(x) < m\}$$
>
> has measure zero, concluding.

P2. Let $(X, \mathscr{B}, \mu)$ be a probability space, $\mathscr{A} \subseteq \mathscr{B}$ be a σ-algebra and $f \in L^1(X, \mathscr{B}, \mu)$. Show that

$$|\mathbb{E}(f \mid \mathscr{A})(x)| \leq \mathbb{E}(|f| \mid \mathscr{A})(x)$$

for almost every $x \in X$.

> By splitting f into its real and imaginary parts, we may assume that it is real-valued. Let $A \in \mathscr{A}$ be the set in which the desired inequality fails, and let $A_+ = \{x \in A \mid \mathbb{E}(f \mid \mathscr{A}) > 0\}$

and $A_- = \{x \in A \mid \mathbb{E}(f \mid \mathscr{A}) \leq 0\}$. Assume by contradiction that $\mu(A) > 0$. Then we have

$$\begin{aligned}\int_A |\mathbb{E}(f \mid \mathscr{A})\ d\mu &= \int_{A_+} |\mathbb{E}(f \mid \mathscr{A})|\ d\mu + \int_{A_-} |\mathbb{E}(f \mid \mathscr{A})|\ d\mu \\ &= \left|\int_{A_+} \mathbb{E}(f \mid \mathscr{A})\ d\mu\right| + \left|\int_{A_-} \mathbb{E}(f \mid \mathscr{A})\ d\mu\right| \\ &= \left|\int_{A_+} f\ d\mu\right| + \left|\int_{A_-} f\ d\mu\right| \\ &\leq \int_{A_+} |f|\ d\mu + \int_{A_-} |f|\ d\mu \\ &= \int_{A_+} \mathbb{E}(|f| \mid \mathscr{A})\ d\mu + \int_{A_-} \mathbb{E}(|f| \mid \mathscr{A})\ d\mu \\ &= \int_A \mathbb{E}(|f| \mid \mathscr{A})\ d\mu < \int_A |\mathbb{E}(f \mid \mathscr{A})|\ d\mu,\end{aligned}$$

which is a contradiction, concluding.

P3. Let $(X, \mathscr{A}, \mu)$ be a probability space, $\mathscr{B}$ a sub-σ-algebra of $\mathscr{A}$, and $x \to \mu_x$ the disintegration of μ with respect $\mathscr{B}$.

(a) Prove that for all $A \in \mathscr{B}$, $\mu_x(A) = \mathbb{1}_A(x)$, for μ-almost every $x \in X$.

Let $A \in \mathscr{B}$, notice that $x \to \mu_x(A)$ and $x \to \mathbb{1}_A(x)$ are $\mathscr{B}$-measurable. Thus, it suffices to prove that for every $B \in \mathscr{B}$,

$$\int_B \mu_x(A)\ d\mu(x) = \int_B \mathbb{1}_A(x)\ d\mu(x).$$

By definition of conditional expectation we have that

$$\int_B \mu_x(A)\ d\mu(x) = \int_B \mathbb{E}(\mathbb{1}_A \mid \mathscr{B})(x)\ d\mu(x) = \int_B \mathbb{1}_A\ d\mu = \mu(A \cap B).$$

(b) Prove that the measures μ_x are constant on almost every fiber, namely that for μ-almost every $x \in X$,

$$\mu_y = \mu_x, \quad \text{for } \mu_x\text{-almost every } y \in X.$$

Let $\mathcal{F}$ be a countable family of bounded Borel functions on X that is dense in $L^1(\nu)$ for every probability measure on X. It suffices to prove that for $f \in \mathcal{F}$, for μ-a.e. $x \in X$,

$$\int_X f d\mu_y = \int_X f d\mu_x, \text{ for } \mu_x - \text{almost every } y \in X.$$

For a given $x \in X$, both sides of this relation are functions of y with respect $\mathscr{B}$. Thus it suffices to check that for every $A \in \mathscr{B}$

$$\int_A \int_X f\ d\mu_y d\mu_x(y) = \mu_x(A) \int_X f\ d\mu_x.$$

Again, both sides of the equation are functions of x, measurable with respect $\mathscr{B}$. Thus, it suffices to prove that for $B \in \mathscr{B}$

$$\int_B \int_A \int_X f \; d\mu_y d\mu_x(y) d\mu(x) = \int_B \mu_x(A) \int_X f \; d\mu_x d\mu(x).$$

Notice that by definition of conditional expectation applied twice

$$\begin{aligned}
\int_B \int_A \int_X f \; d\mu_y d\mu_x(y) d\mu(x) &= \int_B \int_X \left(\mathbb{1}_A(y) \int_X f \; d\mu_y \right) d\mu_x(y) d\mu(x) \\
&= \int_B \mathbb{1}_A(x) \int_X f \; d\mu_x d\mu(x) \\
&= \int_{A \cap B} \int_X f \; d\mu_x d\mu(x) \\
&= \int_{A \cap B} f \; d\mu(x).
\end{aligned}$$

Meanwhile, the right side:

$$\begin{aligned}
\int_B \mu_x(A) \int_X f \; d\mu_x d\mu(x) &= \int_B \mathbb{1}_A(x) \mathbb{E}(f \mid \mathscr{B})(x) \; d\mu(x) \\
&= \int_{A \cap B} f \; d\mu(x),
\end{aligned}$$

where we used the previous part in the first equality.

(c) Now suppose that $(X, \mathscr{A}, \mu, T)$ is an invertible measure-preserving system and that $\mathscr{B}$ is a T-invariant sub-σ-algebra. Prove that

$$T\mu_x = \mu_{Tx},$$

for μ-almost every $x \in X$.

The result follows from the definition of conditional expectation, namely the relationship

$$\int_B f \; d\mu = \int_B \mathbb{E}(f \mid \mathscr{B}) \; d\mu(x),$$

the invariance of $\mathscr{B}$ under T, and the invariance of μ under T imply that

$$\int_B f \circ T \; d\mu = \int_B \mathbb{E}(f \mid \mathscr{B}) \circ T \; d\mu(x),$$

which simultaneously implies that $\mathbb{E}(f \circ T \mid \mathscr{B}) = \mathbb{E}(f \mid \mathscr{B}) \circ T$ μ-a.e..