Ergodic Theory
Solutions to Problem Sheet 12

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P1. Let $(X, \mathscr{A}, \mu, T)$ be a measure-preserving system, and let ξ and η be countable partitions of the system with finite entropy.

(a) Show that $h(T, \xi) \leq H(\xi)$.

We have $\forall n \geq 1$

$$\frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i}\xi) \leq \frac{1}{n} \sum_{i=0}^{n-1} H(T^{-i}\xi) = \frac{1}{n} \sum_{i=0}^{n-1} H(\xi) = H(\xi).$$

Let $n \to \infty$, and we are done.

(b) Show that $h(T, \xi \vee \eta) \leq h(T, \xi) + h(T, \eta)$.

This follows immediately from subadditivity of entropy of partitions. Namely, we have $\forall n \geq 1$

$$\frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i}(\xi \vee \eta)) \leq \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i}\xi) + \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i}\eta).$$

Let $n \to \infty$, and we are done.

(c) Show that $h(T,\eta) \leq h(T,\xi) + H(\eta \,|\, \xi)$.

This property follows by Theorem 116 from the Notes.

$$\begin{aligned} h(T,\eta) &= \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} T^{-i}\eta\Big) \\ &\leq \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} T^{-i}(\xi \vee \eta)\Big) \\ &= \lim_{n\to\infty} \Big[\frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big) + \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} T^{-i}\eta \,\Big|\, \bigvee_{i=0}^{n-1} T^{-i}\xi\Big)\Big] \\ &\leq h(T,\xi) + \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} H\Big(T^{-i}\eta \,|\, T^{-i}\xi\Big) \\ &= h(T,\xi) + H(\eta \,|\, \xi), \end{aligned}$$

as desired, where the last equality comes from the invariance of entropy of partitions under measure-preserving transformations.

(d) Show that $h(T,\xi) = h(T, \bigvee_{i=0}^{k} T^{-i}\xi)$, $\forall k \geq 1$.

For any $k \geq 1$, we obtain

$$\begin{aligned} h\Big(T, \bigvee_{i=0}^{k} T^{-i}\xi\Big) &= \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{j=0}^{n-1}\Big(\bigvee_{i=0}^{k} T^{-i}\xi\Big)\Big) \\ &= \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{k+n-1} T^{-i}\xi\Big) \\ &= \lim_{n\to\infty} \Big(\frac{k+n}{n}\Big) \frac{1}{k+n} H\Big(\bigvee_{i=0}^{k+n-1} T^{-i}\xi\Big) \\ &= h(T,\xi), \end{aligned}$$

as desired.

(e) For T invertible show that $h(T,\xi) = h(T^{-1},\xi) = h(T, \bigvee_{i=-k}^{k} T^{-i}\xi)$, $\forall k \geq 1$.

The second equality follows in very similar fashion to the previous exercise, so we omit the details. We now prove the first equality. By the invariance of entropy of partitions under measure-preserving transformations, we have

$$H\Big(\bigvee_{i=0}^{n-1} T^{i}\xi\Big) = H\Big(T^{-(n-1)} \bigvee_{i=0}^{n-1} T^{i}\xi\Big) = H\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big).$$

The claims follows by dividing by n and taking the limit as $n \to \infty$.

(f) Show that $h(T^k) = kh(T)$, $\forall k \geq 1$.

Let ξ and η be partitions with finite entropy. Then, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{j=0}^{n-1}T^{-kj}\left(\bigvee_{i=0}^{k-1}T^{-i}\xi\right)\right) &= \lim_{n\to\infty}\frac{kn}{n}\frac{1}{kn}H\left(\bigvee_{i=0}^{kn-1}T^{-i}\xi\right)\\ &= kh(T,\xi).\end{aligned}$$

This implies that $h(T^k, \bigvee_{i=0}^{k-1} T^{-i}\xi) = kh(T,\xi)$. Therefore, it follows that

$$h(T^k) \geq kh(T).$$

For the reverse inequality, it is sufficient to note that

$$h(T^k,\eta) \leq h(T^k, \bigvee_{i=0}^{k-1} T^{-i}\eta) = kh(T,\eta),$$

and we are done.

(g) Let $(Y, \mathscr{B}, \nu, S)$ be a measure-preserving system and suppose that $(Y, \mathscr{B}, \nu, S)$ is a factor of $(X, \mathscr{A}, \mu, T)$. Show that $h(S) \leq h(T)$.

Let $\phi : X \mapsto Y$ be the factor map. Then, any partition ξ of Y defines the partition $\phi^{-1}(\xi)$ of X. Since ϕ preserves measure we have

$$H_Y(\xi) = H_X(\phi^{-1}(\xi)),$$

where we use the subscripts to distinguish with respect to which system we are considering entropy. From this it follows that $h(T, \phi^{-1}(\xi)) = h(S,\xi)$, and we obtain the claim.

P2. Let $(X, \mathscr{B}, \mu, T)$ be an invertible measure-preserving system, such that ξ is a generating partition of the system. Show that the entropy of T is 0.

By definition, we have that

$$\mathscr{B} = \sigma\left(\bigvee_{i=0}^{\infty} T^{-i}\xi\right)$$

Using the Kolmogorov-Sinai theorem and by conditioning on the future (Theorem 119 from the Notes) we obtain

$$h(T) = h(T,\xi) = \lim_{n\to\infty} H\left(\xi \,\middle|\, \bigvee_{i=1}^{n} T^{-i}\xi\right).$$

Since T is invertible, let us consider the partition $T\xi$, which has finite entropy. Then, by using the continuity lemma from the notes (Lemma 121) and the invariance of entropy of partitions under measure-transforming transformations, we obtain

$$h(T) = \lim_{n\to\infty} H\left(T\xi \,\middle|\, \bigvee_{i=0}^{n-1} T^{-i}\xi\right) = 0,$$

as desired.

P3. Given $\alpha \in \mathbb{R}$, calculate the entropy of the circle rotation $R_\alpha : \mathbb{T} \mapsto \mathbb{T}$.

We consider two cases. First $\alpha \in \mathbb{Q}$, for which we have that there exists $k \in \mathbb{N}$ such that $T^k = T$ and thus, since the identity map has 0 entropy, it follows that the entropy of the system is 0. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we consider the partition $\xi = \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$. Irrational rotations are dense, and so countably many $R_\alpha^{-i}\xi$ generate any open interval, and since any Borel measurable set is generated by countably many open intervals, it follows that ξ is a generating partition of the system. Therefore, the conclusion follows from the previous problem.

P4. Calculate the entropy of the map $T_d : x \to dx \pmod 1$ on $\mathbb{T}$, for any $d \geq 2$.

We consider the partition $\xi = \{[\frac{a}{d}, \frac{a+1}{d} : a = 0, 1, ..., d-1\}$ and observe that for any $n \in \mathbb{N}$, $\bigvee_{i=0}^{n-1} T_d^{-i}\xi = \{[\frac{a}{d^n}, \frac{a+1}{d^n}) : a = 0, 1, ..., d^n - 1\}$. The rational numbers in $[0, 1)$ with powers of d as denominators are dense in $[0, 1)$. Therefore, ξ is a generating partition of the system. By usingKolmogorov-Sinai's Theorem, we have

$$h(T_d) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T_d^{-i}\xi\right).$$

On the other hand, notice that $\bigvee_{i=0}^{n-1} T_d^{-i}\xi$ consists of d^n intervals of measure $1/d^n$. Therefore, by Theorem 116 from the lecture notes we have that

$$h(T_d) = \lim_{n\to\infty} \frac{1}{n} \log(d^n) = \log(d).$$