Ergodic Theory
Problem Sheet 11

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P1. In this exercise we verify several properties concerning entropy of partitions. To that end, let $(X, \mathscr{A}, \mu)$ be a probability space, and let $\xi = \{A_1, A_2, \dots\}$ and $\eta = \{B_1, B_2, \dots\}$ be a partition of our space.

- **(a)** Prove that if ξ is a finite partition with r atoms, then $H(\xi) \le \log r$, with equality if and only if $\mu(A) = \frac{1}{r}$ for any $A \in \xi$.
- **(b)** Prove that $H(\xi \vee \eta) = H(\eta) + H(\xi \mid \eta)$.
- **(c)** Prove that $H(\xi) \ge H(\xi \mid \eta)$.
- **(d)** Prove that ξ and η are independent if and only if $H(\xi \mid \eta) = H(\xi)$.

P2. Prove that $d(\xi, \eta) = H(\xi \mid \eta) + H(\eta \mid \xi)$ defines a metric in the space of finite partitions (up to sets of measure 0).

P3. Prove that for every $\epsilon > 0$ there is $\delta > 0$ such that if $\xi = \{A_1, A_2, \dots, A_r\}$ and $\eta = \{B_1, B_2, \dots, B_r\}$ are two finite partitions with $\sum_{i=1}^{r} \mu(A_i \Delta B_i) < \delta$, then $d(\xi, \eta) < \epsilon$, where d is the metric defined in the previous problem.

P4. Calculate the entropy of any periodic system.