Ergodic Theory
Solutions to Problem Sheet 10

Course Instructor: Florian K. Richter
Problems by: Jovan Andreevski

November 24, 2025

For any questions or corrections, please send an email to: jovan.andreevski@epfl.ch!

P1. Let $(X, \mathscr{B}, \mu, T)$ be a measure preserving system and $f \in L^2(X)$. We say that f is a **compact function** if $\overline{\{U_T^n f \mid n \in \mathbb{N}\}}$ is compact as a subset of L^2 with the standard norm topology. We denote the set of all compact functions by $\mathcal{H}_{com}$.

(a) Let $\varphi : L^2(X) \times L^2(X) \to L^2(X)$ be a uniformly continuous map that commutes with T, i.e. $\varphi(Tf, Tg) = T\varphi(f, g)$. Prove that if $f, g \in \mathcal{H}_{com}$, then $\varphi(f, g) \in \mathcal{H}_{com}$.

Let $f, g \in \mathcal{H}_{com}$ and fix $\varepsilon > 0$.We want to show that $\varphi(f, g) \in \mathcal{H}_{com}$ or equivalently, we want to show that

$$\{U_T^n \varphi(f, g) \mid n \in \mathbb{N}\}$$

is totally bounded in $L^2(X)$.

By uniform continuity, let $\delta > 0$ be such that if $f_1, f_2, g_1, g_2 \in L^2(X)$ and $||f_1 - f_2|| < \delta$, $||g_1 - g_2|| < \delta$, then $||\varphi(f_1, g_1) - \varphi(f_2, g_2)|| < \varepsilon$.

Since $f \in \mathcal{H}_{com}$, by definition $\overline{\{U_T^n f \mid n \in \mathbb{N}\}}$ is compact in L^2, and hence it is totally bounded. The same holds for g. Therefore, let $\{W_i\}$ and $\{V_i\}$ be finite covers of the orbit closures of f and g, respectively, by balls with diameter less than δ. In particular, for all n, there exist $i(n), j(n)$ such that $T^n f \in W_{i(n)}$ and $T^n g \in V_{j(n)}$.

Since φ commutes with T, it follows that $T^n \varphi(f, g) = \varphi(T^n f, T^n g) \in \varphi(W_i, V_j)$, where $\varphi(W_i, V_j) = \{\varphi(w, v) \mid w \in W_i,\, v \in V_j\}$. By construction, $\varphi(W_i, V_j)$ has diameter at most ε, and moreover, $\{U_T^n \varphi(f, g) \mid n \in \mathbb{N}\} \subseteq \bigcup_{i,j} \varphi(W_i, V_j)$. This implies that the orbit of $\varphi(f, g)$ is contained in the union of finitely many sets with diameter at most ε, and thus, it is totally bounded. Hence, $\varphi(f, g) \in \mathcal{H}_{com}$, as desired. □

Now, recall that in the Jacobs-de Leeuw-Glicksberg decomposition we encountered the space

$$\mathcal{H}_{eig} = \overline{\text{span}\{f \in L^2(X) \mid f \text{ is an eigenfunction}\}}.$$

(b) **(Optional)** Prove that $\mathcal{H}_{com} = \mathcal{H}_{eig}$.

First, we need to show that $\mathcal{H}_{com}$ is a closed subspace of $L^2(X)$. By applying part (a) with $\phi(f, g) = f + g$, we see that $\mathcal{H}_{com}$ is a subspace.

To show that it is closed, let $f \in \overline{\mathcal{H}_{com}}$. We need to show that $f \in \mathcal{H}_{com}$. Let $\varepsilon > 0$, and take $g \in \mathcal{H}_{com}$ such that $||f - g|| < \dfrac{\varepsilon}{2}$. Then, as the Koopman operator is an isometry, we also have

$||T^n f - T^n g|| < \frac{\varepsilon}{2}$, $\forall n \in \mathbb{N}$. Since g is compact, its orbit is totally bounded, meaning that there are finitely many points $x_1, \ldots, x_r$ such that the orbit of g is contained in $\bigcup_{i=1}^{r} B_{\frac{\varepsilon}{2}}(x_i)$. By the triangle inequality, it follows that the orbit of f is contained in $\bigcup_{i=1}^{r} B_{\varepsilon}(x_i)$, and since ε was arbitrary, it follows that f is compact, as desired.

Notice that every eigenfunction is trivially compact, and so, since $\mathcal{H}_{com}$ is a closed subspace, it follows that $\mathcal{H}_{eig} \subseteq \mathcal{H}_{com}$.

Showing the reverse inclusion is the hard part of this exercise. We proceed as follows. Let $f \in \mathcal{H}_{com}$. By definition, $Y = \overline{\{U_T^n f \mid n \in \mathbb{N}\}} \subset L^2(X)$ is compact and invariant under the Koopman operator. Since the Koopman operator is an isometry, the pair (Y, U_T) defines a compact metric space endowed with an isometric homeomorphism. Moreover, it is topologically transitive as it contains a dense orbit. Therefore, (Y, U_T) is a transitive, isometric topological system, and so, a standard result from topological dynamics says that we can endow Y with the structure of a compact abelian group. More precisely,we define the group law as follows: for $g, h \in Y$, suppose that $U_T^{n_i} f \to g$, $U_T^{m_i} f \to h$. Then, $g \cdot h := \operatorname{lím} U_T^{n_i + m_i} f$. Moreover, we can take f to be the identity and $\phi : Y \mapsto Y$ to be the rotation by the element $U_T f \in Y$.

So, we can think of Y as a compact abelian group. Let ν be the Haar measure on Y. For each character χ of Y, define the function $f_\chi = \int_Y \chi(y) y \, \mathrm{d}\nu(y)$. It can be easily checked that f_χ is an eigenfunction for the Koopman operator, and more generally, for any function $\xi \in L^\infty(X)$, we define the function $f_\xi = \int_Y \xi(y) y \, \mathrm{d}\nu(y)$.

Since we can approximate any L^∞ function by finite linear combinations of characters, we can take ξ to be the normalized indicator function of $B_r(f)$, where $r > 0$ is chosen sufficiently small. If we let $r \to 0$, then it is evident that $f_\xi \to f$ in L^2. Therefore, we can find a sequence of finite linear combinations of eigenfunctions which converges to f in L^2, and so $f \in \mathcal{H}_{eig}$. $\square$

In the course we showed that $(X, \mathscr{B}, \mu, T)$ is a Kronecker system if and only if it is isomorphic to an ergodic group rotation.

The system $(X, \mathscr{B}, \mu, T)$ has a discrete spectrum exactly when $L^2(X) = \mathcal{H}_{eig}$ (**Why?**).

Therefore, having in mind part (b), another important equivalent characterisation of Kronecker systems is that $(X, \mathscr{B}, \mu, T)$ is a Kronecker system if and only if $(X, \mathscr{B}, \mu, T)$ is a **compact system**, i.e. every $f \in L^2(X)$ is a compact function.

(c) Use this equivalent characterisation of Kronecker systems to prove the following result: Let $(X, \mathscr{B}, \mu, T)$ be a Kronecker system, let $k \in \mathbb{N}$ and $f \in L^\infty(X)$ be real-valued. Then, $\forall \varepsilon > 0$, the following set is syndetic:

$$\left\{ n \in \mathbb{N} \,\middle|\, \int_X \prod_{i=0}^{k} U_T^{ni} f \mathrm{d}\mu > \int_X f^{k+1} \mathrm{d}\mu - \varepsilon \right\}.$$

You may use the following standard lemma:

Lemma 1. *Let X be a compact metric space, $T : X \mapsto X$ an isometry, and let $x \in X$. Then, for any open neighborhood $U \subset X$ of x, the set $\{n \in \mathbb{N} \mid T^n x \in U\}$ is syndetic.*

[**Hint:** Show that the set in question is a superset of a syndetic set.]

By rescaling we may assume that $||f||_\infty \leq 1$. Since $(X, \mathscr{B}, \mu, T)$ is a Kronecker system, by the equivalent characterisation, it is a compact system, and so f is a compact function. Therefore, by applying Lemma 1 with $\overline{\{U_T^n f \mid n \in \mathbb{N}\}}$, the point $f \in \overline{\{U_T^n f \mid n \in \mathbb{N}\}}$ and the

open ball around f of radius $\frac{\varepsilon}{k^2}$, it follows that the set $M = \{n \in \mathbb{N} \,|\, ||T^n f - f|| < \frac{\varepsilon}{k^2}\}$ is syndetic.
Using the triangle inequality and a telescoping sum, and recalling that U_T^n is an isometry, we notice that $\forall n \in M$ and $i = 0, \dots, k$, we have

$$\begin{aligned} ||T^{in} f - f|| \leq ||T^{in} f - T^{(i-1)n} f|| + ||T^{(i-1)n} f - T^{(i-2)n} f|| + \dots + ||T^n f - f|| &= i||T^n f - f|| \\ &< i\frac{\varepsilon}{k^2} \\ &\leq \frac{\varepsilon}{k}. \end{aligned}$$

Now, take $I_0(n) = \int_X \prod_{i=0}^k U_T^{ni} f \mathrm{d}\mu = \int_X f \cdot \prod_{i=1}^k U_T^{ni} f \mathrm{d}\mu$ and $I_1(n) = \int_X f^2 \cdot \prod_{i=2}^k U_T^{ni} f \mathrm{d}\mu$. Observe that

$$I_0(n) - I_1(n) = \int_X f \cdot (U_T^n f - f) \cdot \prod_{i=2}^k U_T^{ni} f \mathrm{d}\mu$$

Since $||f||_\infty \leq 1$ and $||T^{in} f - f|| < \frac{\varepsilon}{k}$ for all $i \in \{0, \dots, k\}$, by applying the Cauchy Schwarz inequality with $f \cdot \prod_{i=2}^k U_T^{ni} f$ and $U_T^n f - f$, we obtain

$$\begin{aligned} |I_0(n) - I_1(n)| \leq \frac{\varepsilon}{k} &\implies I_0(n) \geq I_1(n) - \frac{\varepsilon}{k} \\ &\iff \int_X f \cdot \prod_{i=1}^k U_T^{ni} f \mathrm{d}\mu \geq \int_X f^2 \cdot \prod_{i=2}^k U_T^{ni} f \mathrm{d}\mu - \frac{\varepsilon}{k} \end{aligned}$$

By repeatedly applying the Cauchy-Schwarz inequality in the same manner with $I_j(n) = \int_X f^{j+1} \cdot \prod_{i=j+1}^k U_T^{ni} f \mathrm{d}\mu$, we obtain that $\forall n \in M$

$$\begin{aligned} \int_X \prod_{i=0}^k U_T^{ni} f \mathrm{d}\mu = \int_X f \cdot \prod_{i=1}^k U_T^{ni} f \mathrm{d}\mu \geq \int_X f^2 \cdot \prod_{i=2}^k U_T^{ni} f \mathrm{d}\mu - \frac{\varepsilon}{k} &\geq \int_X f^3 \cdot \prod_{i=3}^k U_T^{ni} f \mathrm{d}\mu - \frac{2\varepsilon}{k} \\ &\geq \dots \geq \int_X f^{k+1} \mathrm{d}\mu - \varepsilon. \end{aligned}$$

This shows that $M \subseteq \left\{ n \in \mathbb{N} \,|\, \int_X \prod_{i=0}^k U_T^{ni} f \mathrm{d}\mu > \int_X f^{k+1} \mathrm{d}\mu - \varepsilon \right\}$, and since M is syndetic, this concludes the proof. □

P2. The purpose of the previous exercise is to combine it with the Jacobs-de Leeuw-Glicksberg decomposition and prove a celebrated result from Combinatorial Number Theory due to Roth:

Theorem 1 (Roth, 1953). *Let $A \subset \mathbb{N}$ have positive upper density, i.e.*

$$\bar{d}(A) := \limsup_{N \to \infty} \frac{|A \cap [1, N]|}{N} > 0. \tag{1}$$

Then, A contains a 3-term arithmetic progression.

A famous result in Ergodic Ramsey Theory due to Furstenberg (1977), known as **Furstenberg's Correspondence Principle**, allows us to reduce Roth's Theorem to the following result about measure preserving systems:

Theorem 2. *Let $(X, \mathscr{B}, \mu, T)$ be an ergodic measure-preserving system and let $A \in \mathscr{B}$ such that $\mu(A) > 0$. Then, $\exists n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}A \cap T^{-2n}A) > 0$. In fact, we have*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) > 0. \tag{2}$$

The aim is to prove Theorem 2, and thus Theorem 1.

(a) Let $(X, \mathscr{B}, \mu, T)$ be a measure preserving system, let $f \in L^2(X)$ and let $f = f_{com} + f_{wm}$ be the Jacobs-de Leeuw-Glicksberg decomposition of f. Show that if f takes values in $[0, 1]$, then so does f_{com}.
[**Hint:** Use part (a) of exercise 1 with the functions $\min(f, g)$ and $\max(f, g)$.]

> Let $f_0 = \Re f_{com}$ denote the real part of f_{com}. Since the orbit closure of f_0 is the real part of the orbit closure of f_{com}, and $\Re : L^2 \to L^2$ is a continuous map, it follows that f_0 is compact.
> $\mathcal{H}_{com}$ contains the constant functions, and so by part (a) of Exercise 1, $f_1 = \min(f_0, 1) \in \mathcal{H}_{com}$ and consequently $f_2 = \max(f_1, 0) \in \mathcal{H}_{com}$. Notice that f_2 takes values in $[0, 1]$, and moreover, $||f_0 - f|| \leq ||f_{com} - f||$. This further implies that
>
> $$||f_2 - f|| \leq ||f_1 - f|| \leq ||f_0 - f|| \leq ||f_{com} - f||.$$
>
> By the Jacobs-de Leeuw-Glicksberg decomposition $f - f_{com} \in \mathcal{H}_{wm}$, and so $f - f_{com} \perp \mathcal{H}_{com}$, which in particular means $\langle f - f_{com}, f_{com} - f_2 \rangle = 0$. Therefore,
>
> $$||f - f_{com}||^2 \geq ||f - f_2||^2 = ||f - f_{com} + f_{com} - f_2||^2 = ||f - f_{com}||^2 + ||f_{com} - f_2||^2.$$
>
> This implies that $||f_{com} - f_2||^2 = 0$, which means that $f_{com} = f_2 \in [0, 1]$, as desired. □

(b) **(Optional)** Prove van der Corput's lemma: let $\mathcal{H}$ be a Hilbert space and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence taking values in $\mathcal{H}$. Show that if

$$\lim_{H\to+\infty} \frac{1}{H} \sum_{0\leq h\leq H} \limsup_{N\to+\infty} \left| \frac{1}{N} \sum_{1\leq n\leq N} \langle u_{n+h}, u_n \rangle \right| = 0,$$

then

$$\lim_{N\to+\infty} \left\| \frac{1}{N} \sum_{1\leq n\leq N} u_n \right\| = 0.$$

Note that the same statement holds for uniform Cesàro averages.

> By boundedness, we can rescale the sequence, so that $\|u_n\| \leq 1$. We prove the quantitatively precise and more useful estimate
>
> $$\left\| \frac{1}{N} \sum_{n\leq N} u_n \right\|^2 \leq \frac{C_1}{H} \sum_{h\leq H} \left| \frac{1}{N} \sum_{n\leq N} \langle u_{n+h}, u_n \rangle \right| + \frac{C_2}{H} + \frac{C_3 H}{N}$$
>
> valid for any $1 \leq H \leq N$ and some absolute constants C_1, C_2, C_3 that will be calculated throughout the proof (the constants are unimportant in general). The statement follows

easily by taking $N \to +\infty$ and then $H \to +\infty$. For each $h \in [1, H] \cap \mathbb{Z}$, we have

$$\left\| \frac{1}{N} \sum_{n \le N} u_n - \frac{1}{N} \sum_{n \le N} u_{n+h} \right\| \le \frac{h}{N} \le \frac{H}{N}$$

and averaging over all $h \le H$, we get that

$$\left\| \frac{1}{N} \sum_{n \le N} u_n - \frac{1}{H} \sum_{h \le H} \frac{1}{N} \sum_{n \le N} u_{n+h} \right\| \le \frac{H}{N} \tag{3}$$

Applying the triangle inequality and then Cauchy-Schwarz, we conclude

$$\begin{aligned} \left\| \frac{1}{H} \sum_{h \le H} \frac{1}{N} \sum_{n \le N} u_{n+h} \right\|^2 &= \left\| \frac{1}{N} \sum_{n \le N} \frac{1}{H} \sum_{h \le H} u_{n+h} \right\|^2 \\ &\le \left(\frac{1}{N} \sum_{n \le N} \left\| \frac{1}{H} \sum_{h \le H} u_{n+h} \right\| \right)^2 \le \frac{1}{N} \sum_{n \le N} \left\| \frac{1}{H} \sum_{h \le H} u_{n+h} \right\|^2. \end{aligned} \tag{4}$$

We expand out the right-hand side as

$$\frac{1}{N} \sum_{n \le N} \frac{1}{H^2} \sum_{1 \le h_1, h_2 \le H} \langle u_{n+h_1}, u_{n+h_2} \rangle$$

The diagonal contribution $h_1 = h_2$ is equal to

$$\frac{1}{N} \sum_{n \le N} \frac{1}{H^2} \sum_{h \le H} \|u_{n+h}\|^2 \le \frac{1}{H},$$

since all terms are bounded in magnitude by 1.
For the off-diagonal contribution, we use symmetry to get that

$$\left| \frac{1}{N} \sum_{n \le N} \frac{1}{H^2} \sum_{1 \le h_1, h_2 \le H} \langle u_{n+h_1}, u_{n+h_2} \rangle \right| \le \left| \frac{2}{N} \sum_{n \le N} \frac{1}{H^2} \sum_{1 \le h_1 < h_2 \le H} \langle u_{n+h_1}, u_{n+h_2} \rangle \right| =$$
$$\left| \frac{2}{H^2} \sum_{1 \le h_1 < h_2 \le H} \frac{1}{N} \sum_{n \le N} \langle u_{n+h_1}, u_{n+h_2} \rangle \right| \le \frac{2}{H^2} \sum_{1 \le h_1 < h_2 \le H} \left| \frac{1}{N} \sum_{n \le N} \langle u_{n+h_1}, u_{n+h_2} \rangle \right|.$$

Observe that

$$\left| \frac{1}{N} \sum_{n \le N} \langle u_{n+h_1}, u_{n+h_2} \rangle \right| = \left| \frac{1}{N} \sum_{h_1 + n \le N + h_1} \langle u_n, u_{n+h_2-h_1} \rangle \right| \le$$
$$\left| \frac{1}{N} \sum_{n \le N+h_1} \langle u_n, u_{n+h_2-h_1} \rangle \right| + \frac{2h_1}{N} \le \left| \frac{1}{N} \sum_{n \le N} \langle u_n, u_{n+h_2-h_1} \rangle \right| + \frac{2H}{N},$$

where we used the triangle inequality for the first inequality and bounded trivially each term by 1.

Using this inequality, we get that the off-diagonal contributions is bounded by

$$\frac{2}{H^2} \sum_{1 \le h_1 < h_2 \le H} \left(\left| \frac{1}{N} \sum_{n \le N} \langle u_n, u_{n+h_2-h_1} \rangle \right| + \frac{2H}{N} \right) \le$$

$$\frac{2}{H^2} \sum_{1 \le h_1 < h_2 \le H} \left| \frac{1}{N} \sum_{n \le N} \langle u_n, u_{n+h_2-h_1} \rangle \right| + \frac{2H}{N}$$

The difference $h_2 - h_1$ takes values in $1, \ldots H-1$ and a simple counting yields that every r in this range occurs as the difference of $H - r$ pairs. Therefore, we may rewrite the right-hand side as

$$\frac{2}{H^2} \sum_{1 \le r \le H-1} (H-r) \left| \frac{1}{N} \sum_{n \le N} \langle u_n, u_{n+r} \rangle \right| + \frac{2H}{N} \le \frac{2}{H} \sum_{1 \le r \le H-1} \left| \frac{1}{N} \sum_{n \le N} \langle u_n, u_{n+r} \rangle \right| + \frac{2H}{N} \le$$

$$\frac{2}{H} \sum_{1 \le r \le H} \left| \frac{1}{N} \sum_{n \le N} \langle u_n, u_{n+r} \rangle \right| + \frac{2H}{N} + \frac{2}{H}.$$

Using this, as well as the contribution from the diagonal terms, we deduce from (4) the bound

$$\left\| \frac{1}{H} \sum_{h \le H} \frac{1}{N} \sum_{n \le N} u_{n+h} \right\|^2 \le \frac{2}{H} \sum_{1 \le r \le H} \left| \frac{1}{N} \sum_{n \le N} \langle u_n, u_{n+r} \rangle \right| + \frac{2H}{N} + \frac{3}{H}.$$

Using (3) and the triangle inequality, we deduce that

$$\left\| \frac{1}{N} \sum_{n \le N} u_n \right\|^2 \le \frac{2}{H} \sum_{h \le H} \left| \frac{1}{N} \sum_{n \le N} \langle u_n, u_{n+h} \rangle \right| + 3 \left(\frac{H}{N} + \frac{1}{H} \right).$$

(c) Let $(X, \mathscr{B}, \mu, T)$ be an ergodic measure preserving system, and let $f, g \in L^2(X)$. Show that if f or g (or both) is weak-mixing, then

$$\lim_{N-M \to \infty} \frac{1}{N-M} \sum_{n=M}^{N} T^n f \cdot T^{2n} g = 0$$

in norm.

[**Hint:** Use van der Corput's Lemma for uniform Cesàro averages with the sequence $u_n = T^n f \cdot T^{2n} g$.]

Following the hint, we first obtain

$$\langle u_{n+h}, u_n \rangle = \int_X T^{n+h}f \cdot T^{2n+2h}g \cdot T^n\overline{f} \cdot T^{2n}\overline{g}\,\mathrm{d}\mu = \int_X T^n(T^hf \cdot \overline{f}) \cdot T^{2n}(T^{2h}g \cdot \overline{g})\,\mathrm{d}\mu$$
$$= \int_X (T^hf \cdot \overline{f}) \cdot T^n(T^{2h}g \cdot \overline{g})\,\mathrm{d}\mu$$
$$= \int_X F_h \cdot T^nG_h\,\mathrm{d}\mu,$$

where we set $F_h = T^hf \cdot \overline{f}$, $G_h = T^{2h}g \cdot \overline{g}$. Using ergodicity and the Mean Ergodic theorem, by taking a uniform Cesàro average in n, we obtain

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} \langle u_{n+h}, u_n \rangle = \int_X F_h\,\mathrm{d}\mu \int_X G_h\,\mathrm{d}\mu.$$

Since both sequences $h \mapsto \int_X F_h\,\mathrm{d}\mu$ and $h \mapsto \int_X G_h\,\mathrm{d}\mu$ are bounded by the Cauchy-Schwarz inequality, and since $\forall\varepsilon > 0$, the one associated with the weak-mixing function is smaller than ε in a set of full density (this follows by the definition of weak-mixing), we obtain

$$\limsup_{H\to+\infty} \frac{1}{H}\sum_{0\leq h\leq H} \limsup_{N\to+\infty} \left| \frac{1}{N-M}\sum_{n=M}^{N} \langle u_{n+h}, u_n \rangle \right| < \varepsilon.$$

for every $\varepsilon > 0$. Therefore, we can apply van der Corput's lemma to obtain that

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} T^nf \cdot T^{2n}g = 0$$

in norm, as desired. □

(d) Let $(X, \mathscr{B}, \mu, T)$ be an ergodic measure-preserving system and let $A \in \mathscr{B}$ such that $\mu(A) > 0$. Prove that

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) > 0.$$

[**Hint:** Use part (c) of exercise 1.]

Our efforts in this exercise sheet come to full fruition in this exercise, where we put everything together and prove Theorem 2. The main idea is to use the Jacobs-de Leeuw-Glicksberg decomposition on the indicator function $\mathbb{1}_A$. To that end, let $\mathbb{1}_A = f_{wm} + f_{com}$, from which we obtain

$$\begin{aligned}\mu(A\cap T^{-n}A\cap T^{-2n}A) &= \int_X \mathbb{1}_A\cdot T^n\mathbb{1}_A\cdot T^{2n}\mathbb{1}_A\,\mathrm{d}\mu\\ &= \int_X \mathbb{1}_A\cdot T^n(f_{wm}+f_{com})\cdot T^{2n}(f_{wm}+f_{com})\,\mathrm{d}\mu\\ &= \int_X \mathbb{1}_A\cdot T^n f_{wm}\cdot T^{2n}f_{wm}\,\mathrm{d}\mu + \int_X \mathbb{1}_A\cdot T^n f_{wm}\cdot T^{2n}f_{com}\,\mathrm{d}\mu\\ &+ \int_X \mathbb{1}_A\cdot T^n f_{com}\cdot T^{2n}f_{wm}\,\mathrm{d}\mu + \int_X \mathbb{1}_A\cdot T^n f_{com}\cdot T^{2n}f_{com}\,\mathrm{d}\mu.\end{aligned}$$

The strategy for the proof is to show that the last term has a positive average, whereas the other three terms have an average of 0.

By part (a) of Exercise 2, we know that since $f = \mathbb{1}_A \in [0,1]$, $f_{com} \in [0,1]$. Moreover, since the constant functions are compact, we have that $1 \perp f_{wm}$. Therefore,

$$\int_X f_{com}\,\mathrm{d}\mu = \langle f_{com}, 1\rangle = \langle \mathbb{1}_A, 1\rangle = \mu(A) > 0.$$

Therefore, $f_{com} \neq 0$ a.e., and so by applying part (c) of Exercise 1 with $k = 2$, we obtain

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N}\int_X f_{com}\cdot T^n f_{com}\cdot T^{2n}f_{com}\,\mathrm{d}\mu > 0.$$

Since $f_{com} \in \mathcal{H}_{com}$, it is straightforward to see that also $T^n f_{com}\cdot T^{2n}f_{com} \in \mathcal{H}_{com}$, and we obtain that $T^n f_{com}\cdot T^{2n}f_{com} \perp \mathcal{H}_{wm}$. In particular, for all n, $T^n f_{com}\cdot T^{2n}f_{com} \perp f_{wm}$ and thus

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N}\int_X \mathbb{1}_A\cdot T^n f_{com}\cdot T^{2n}f_{com}\,\mathrm{d}\mu > 0. \tag{5}$$

Finally, we apply part (c) from Exercise 2 to the other three terms, and obtain

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N}\int_X \mathbb{1}_A\cdot T^n f_{wm}\cdot T^{2n}f_{wm}\,\mathrm{d}\mu = 0, \tag{6}$$

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N}\int_X \mathbb{1}_A\cdot T^n f_{wm}\cdot T^{2n}f_{com}\,\mathrm{d}\mu = 0, \tag{7}$$

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N}\int_X \mathbb{1}_A\cdot T^n f_{com}\cdot T^{2n}f_{wm}\,\mathrm{d}\mu = 0. \tag{8}$$

Adding (5), (6), (7) and (8), yields the desired result. □