Ergodic Theory
Problem Sheet 10

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November 24, 2025

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P1. Let $(X, \mathscr{B}, \mu, T)$ be a measure preserving system and $f \in L^2(X)$. We say that f is a **compact function** if $\overline{\{U_T^n f \mid n \in \mathbb{N}\}}$ is compact as a subset of L^2 with the standard norm topology. We denote the set of all compact functions by $\mathcal{H}_{com}$.

(a) Let $\varphi : L^2(X) \times L^2(X) \to L^2(X)$ be a uniformly continuous map that commutes with T, i.e. $\varphi(Tf, Tg) = T\varphi(f, g)$. Prove that if $f, g \in \mathcal{H}_{com}$, then $\varphi(f, g) \in \mathcal{H}_{com}$.

Now, recall that in the Jacobs-de Leeuw-Glicksberg decomposition we encountered the space

$$\mathcal{H}_{eig} = \overline{\text{span}\{f \in L^2(X) \mid f \text{ is an eigenfunction}\}}.$$

(b) **(Optional)** Prove that $\mathcal{H}_{com} = \mathcal{H}_{eig}$.

In class we showed that $(X, \mathscr{B}, \mu, T)$ is a Kronecker system if and only if it is isomorphic to an ergodic group rotation.

The system $(X, \mathscr{B}, \mu, T)$ has a discrete spectrum precisely when $L^2(X) = \mathcal{H}_{eig}$ (**Why?**).

Therefore, having in mind part (b), another important equivalent characterisation of Kronecker systems is that $(X, \mathscr{B}, \mu, T)$ is a Kronecker system if and only if $(X, \mathscr{B}, \mu, T)$ is a **compact system**, i.e. every $f \in L^2(X)$ is a compact function.

(c) Use this equivalent characterisation of Kronecker systems to prove the following result: Let $(X, \mathscr{B}, \mu, T)$ be a Kronecker system, let $k \in \mathbb{N}$ and $f \in L^\infty(X)$. Then, $\forall \varepsilon > 0$, the following set is syndetic:

$$\left\{ n \in \mathbb{N} \mid \int_X \prod_{i=0}^{k} U_T^{ni} f \, \mathrm{d}\mu > \int_X f^{k+1} \mathrm{d}\mu - \varepsilon \right\}.$$

You may use the following standard lemma:

Lemma 1. *Let X be a compact metric space, $T : X \mapsto X$ an isometry, and let $x \in X$. Then, for any open neighborhood $U \subset X$ of x, the set $\{n \in \mathbb{N} \mid T^n x \in U\}$ is syndetic.*

[**Hint:** Show that the set in question is a superset of a syndetic set.]

P2. The purpose of the previous exercise is to combine it with the Jacobs-de Leeuw-Glicksberg decomposition and prove a celebrated result from Combinatorial Number Theory due to Roth:

Theorem 1 (Roth, 1953)**.** *Let $A \subset \mathbb{N}$ have positive upper density, i.e.*

$$\bar{d}(A) := \limsup_{N\to\infty} \frac{|A \cap [1,N]|}{N} > 0. \tag{1}$$

Then, A contains a 3-term arithmetic progression.

A famous result in Ergodic Ramsey Theory due to Furstenberg (1977), known as **Furstenberg's Correspondence Principle**, allows us to reduce Roth's Theorem to the following result about measure preserving systems:

Theorem 2. *Let $(X, \mathscr{B}, \mu, T)$ be an ergodic measure-preserving system and let $A \in \mathscr{B}$ such that $\mu(A) > 0$. Then, $\exists\, n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}A \cap T^{-2n}A) > 0$. In fact, we have*

$$\mathit{lím}_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) > 0. \tag{2}$$

The aim is to prove Theorem 2, and thus Theorem 1.

(a) Let $(X, \mathscr{B}, \mu, T)$ be a measure preserving system, let $f \in L^2(X)$ and let $f = f_{com} + f_{wm}$ be the Jacobs-de Leeuw-Glicksberg decomposition of f. Show that if f takes values in $[0,1]$, then so does f_{com}.

[**Hint:** Use part (a) of exercise 1 with the functions $\mathrm{mín}(f,g)$ and $\mathrm{máx}(f,g)$.]

(b) **(Optional)** Prove van der Corput's lemma: let $\mathcal{H}$ be a Hilbert space and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence taking values in $\mathcal{H}$. Show that if

$$\mathrm{lím}_{H\to+\infty} \frac{1}{H} \sum_{0\le h\le H} \limsup_{N\to+\infty} \left| \frac{1}{N} \sum_{1\le n\le N} \langle u_{n+h}, u_n \rangle \right| = 0,$$

then

$$\mathrm{lím}_{N\to+\infty} \left\| \frac{1}{N} \sum_{1\le n\le N} u_n \right\| = 0.$$

Note that the same statement holds for uniform Cesàro averages.

(c) Let $(X, \mathscr{B}, \mu, T)$ be an ergodic measure preserving system, and let $f, g \in L^\infty(X)$. Show that if f or g (or both) is weak-mixing, then

$$\mathrm{lím}_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} T^n f \cdot T^{2n} g = 0$$

in norm.

[**Hint:** Use van der Corput's Lemma for uniform Cesàro averages with the sequence $u_n = T^n f \cdot T^{2n} g$.]

(d) Let $(X, \mathscr{B}, \mu, T)$ be an ergodic measure-preserving system and let $A \in \mathscr{B}$ such that $\mu(A) > 0$. Prove that

$$\mathrm{lím}_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) > 0.$$

[**Hint:** Use part (c) of exercise 1.]