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EPFL, fall semester 2025
AG II - Schemes and sheaves

Solutions – week 9

Exercise 1. *Dual.* Let $\mathcal{E}$ be locally free sheaf of finite rank[1] on a ringed space $(X, \mathcal{O}_X)$ and $\mathcal{F}$ an $\mathcal{O}_X$-module. We define $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$. Show that there is a natural isomorphism $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$.

Solution. Say A is a ring M a finite projective A-module and N any A-module. The key is that the evaluation map

$$M^\vee \otimes_A N \to \operatorname{Hom}_A(M, N)$$

sending a pure tensor $\varphi \otimes n$ to $m \mapsto \varphi(n)m$ is natural in M – we mean by this that if $\psi \colon M \to M'$ then

$$\begin{array}{ccc} M^\vee \otimes_A N & \longrightarrow & \operatorname{Hom}_A(M, N) \\ {\scriptstyle \psi^\vee \otimes \mathrm{id}} \uparrow & & \uparrow {\scriptstyle - \circ \psi} \\ M'^\vee \otimes_A N & \longrightarrow & \operatorname{Hom}_A(M', N) \end{array}$$

commutes. In addition, this evaluation morphism commutes to localization. Therefore, using the above, we can prove the statement at stalks and suppose that $\mathcal{E}_x = \mathcal{O}_{X,x}^n$ for some n. □

Exercise 2. *Compatibilities between f^*, f_* and $\otimes$.* Let $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) Let $\mathcal{G}$ and $\mathcal{H}$ be sheaves of $\mathcal{O}_Y$-modules. Show that there is a natural isomorphism

$$f^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{H}) \cong f^*(\mathcal{G}) \otimes_{\mathcal{O}_X} f^*(\mathcal{H}).$$

(2) (*Projection formula.*) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{E}$ be a finite locally free sheaf on $\mathcal{O}_Y$. Show that there is a natural isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_Y} f_*\mathcal{F} \to f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

Proof. or (1), one may show first that

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes, \mathcal{G}, \mathcal{H}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})).$$

Then the claim follows by combining adjunctions.
For (2), we proceed as in the previous exercise, meaning we construct a natural morphism between the implicit functors in $\mathcal{E}$, and then using this

[1] For $n \in \mathbb{N}$ a locally free sheaf of rank n is an $\mathcal{O}_X$-module which is locally isomorphic to $\mathcal{O}_U^{\oplus n}$ where U ranges in an open cover of X.

naturality we are allowed to show the claim locally. The natural map correspond by adjunction to tensoring the counit map $f^*\mathcal{E} \otimes (f^* f_* \mathcal{F} \to \mathcal{F})$. $\square$

Exercise 3. *Points of projective spaces, naturally.* Let k be a field and V be a finite dimensional vector space. Define

$$\mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}(V^\vee)).$$

Let $l \subset V$ be a k-linear subspace of dimension 1.

(1) Show that if V has dimension 1, then the natural map

$$\mathbb{P}(V) \to \mathrm{Spec}(k)$$

is an isomorphism. Therefore by functoriality of Proj when V is a general finite dimensional vector space, deduce that the map induced by $l \subset V$[2]

$$\mathbb{P}(l) \to \mathbb{P}(V)$$

defines k-rational point.

(2) Moreover, show this defines a bijection

$$\mathbb{P}(V)(k) \leftrightarrow \{l \subset V \mid l \text{ is a one dimensional linear subspace of } V\}.$$

Solution key. If $l \subset V$ is a one dimensional subspace, let $\varphi \in V^\vee$ such that $\varphi(l) \neq 0$. Then we deduce that

$$l^\vee = k\varphi.$$

We therefore have $\mathrm{Sym}(l^\vee) = k[\varphi]$, where φ is treated as a variable. Therefore $\mathrm{Proj}(\mathrm{Sym}(l^\vee)) = \mathrm{Spec}(k)$ because $D_+(\varphi)$ covers $\mathrm{Proj}(\mathrm{Sym}(l^\vee))$ and $\mathrm{Sym}(l^\vee)_{(\varphi)} = k$.
It follows that $V^\vee \to l^\vee$ induces a surjection $\mathrm{Sym}(V^\vee) \to \mathrm{Sym}(l^\vee)$ which induces a closed immersion which is a k-rational point

$$\mathbb{P}(l) \to \mathbb{P}(V)$$

Write $\pi \colon \mathbb{P}(V) \to \mathrm{Spec}(k)$ for the structure map. Say $x \colon \mathrm{Spec}(k) \to \mathbb{P}(V)$ is a k-rational point. Consider the surjection of quasi-coherent sheaves

$$\pi^* V^\vee \to \mathcal{O}(1).$$

This comes from the multiplication map at level of graded modules

$$\mathrm{Sym}(V^\vee) \otimes_A V^\vee \to \mathrm{Sym}(V^\vee)(1),$$

meaning by this that it is sent to the above by the exact functor $\widetilde{-}$. Now take x^* to the above surjection of sheaves. It leads to a surjection

$$V^\vee \to x^*\mathcal{O}(1).$$

Recall that $x^*\mathcal{O}(1)$ is a k-vector space of dimension 1. Up to choosing a generator of it, there is some $v \neq 0$ such that the above map can be seen at the evaluation at v. Note that choosing a different generator would only alter v up to a scalar. So $l := \langle v \rangle$ only depend on the above map. One now checks that the induced $\mathbb{P}(l) \to \mathbb{P}(V)$ gives the other direction of the construction. $\square$

[2] So Proj of the induced map $\mathrm{Sym}(V^\vee) \to \mathrm{Sym}(l^\vee)$.

Remark. The above correspondence can also be seen as follows. To a $l \subset V$, we associate the homogeneous prime ideal of $\mathrm{Sym}(V^\vee)$ generated in degree 1 by the kernel of[3]

$$V^\vee \to l^\vee.$$

From this perspective, we see that $l \in D_+(\varphi)$ if and only if $\varphi(l) \neq 0$.
Choosing a basis, we recover what we are used to. Say $V = k^{n+1}$ and $v = \sum_{i=0}^n \lambda_i e_i$, then the above map is the linear form $(\lambda_0 \cdots \lambda_n)$. The kernel of this map is generated by $(\lambda_j x_i - \lambda_i x_j)$ where x_i denotes the dual basis.

Exercise 4. *Tautological line bundle.*
Let A be a ring and M be a finite projective module. Consider[4]

$$\mathbb{P}(M) := \mathrm{Proj}(\mathrm{Sym}(M^\vee)) \xrightarrow{\pi} \mathrm{Spec}(A).$$

Let $c \in M \otimes M^\vee$ be the canonical element corresponding to the identity along the natural isomorphism $M \otimes M^\vee \cong \mathrm{Hom}_A(M, M)$.

(1) Show that the sub-module generated on $D_+(\varphi)$ by

$$c/\varphi \in \pi^* M(D_+(\varphi)) = \mathrm{Sym}(M^\vee)_{(\varphi)} \otimes M$$

defines a *line bundle* that we denote $\mathcal{O}(-1) \subset \pi^* M$. Show that this line bundle is isomorphic to the one defined in the lecture.

(2) Show that $\mathbb{V}(\mathcal{O}(-1))$ is a closed subscheme of $\mathbb{V}(\pi^* M) = \mathbb{V}(M) \times_A \mathbb{P}(M)$.

(3) *Tautological line bundle.* For any point $x \in \mathrm{Spec}(A)$ show that when base-changing to $\mathrm{Spec}(k(x))$ we obtain the following closed subset on $k(x)$-rational points

$$\mathbb{V}(\mathcal{O}(-1))(k(x)) = \{(v, l) \subset M(x) \times \mathbb{P}(M(x))(k(x)) \mid v \in l\}.$$

Solution key. Note that (2) holds because if $\mathcal{E} \to \mathcal{E}'$ is a sub-vector bundle (see lemmas on vector bundles at the end of this document) then $\mathcal{E}'^\vee \to \mathcal{E}^\vee$ is surjective, which induces a surjective map on Sym, and therefore a closed immersion on relative Spec. That $\mathcal{O}(-1) \subset \pi^* M$ is a sub-vector bundle will follow from the proof below.
Write

$$c = \sum_i x_i \otimes e_i$$

for some elements $x_i \in M^\vee$ and $e_i \in M$.[5] These elements have the property that for any $m \in M$ and $\varphi \in M^\vee$ we have

$$m = \sum_i x_i(m) e_i. \quad \varphi = \sum_i \varphi(e_i) x_i. \tag{1}$$

This will be used at the end of the proof.

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[3]Using exercise 4, one sees that $x^* \mathcal{O}(1)$ naturally identifies with $l^\vee$.

[4]Called the projective bundle associated to M.

[5]If M is free, and (e_i) is a basis, and x_i a dual basis, $c = \sum_i x_i \otimes e_i$.

We slightly differ from the intended route by giving graded constructions. Namely, (1) can be more naturally seen without taking homogeneous localization as the graded inclusion of graded $\mathrm{Sym}(M^\vee)$-graded-modules

$$\langle c\rangle \subset \mathrm{Sym}(M^\vee) \otimes_A M$$

where c is in degree 1. We claim that

$$\mathrm{Sym}(M^\vee)(-1) \to \langle c\rangle$$

sending 1 to c (note that the twist is here to make this map a graded map) is an isomorphism. It is surjective by definition. If the ring A has no-non-zero divisors, then it is automatically an isomorphism. Otherwise, it is clear in the case where M is free – the reader can see how to reduce to this case in a remark at the end of the proof. We now prove (3). We take x in $\mathrm{Spec}(A)$ and base change the setup to $\mathrm{Spec}(k(x))$ (see again the remark after the proof). So we may therefore treat the case where $A = k$ is a field. Then we want to understand k-rational points of $\mathbb{V}(\mathcal{O}(-1))$. The image of such a point along $\mathbb{V}(\mathcal{O}(-1)) \to \mathbb{P}(M)$ is a k-rational of $\mathbb{P}(M)$. By the above exercise, it corresponds to

$$l \subset M$$

a 1-dimensional sub-k-vector space. Say we fix a such. We want to compute the fiber of $\mathcal{O}(-1)$ at this point. Taking first the base change at the level of graded modules we get

$$\langle c_{|l}\rangle \subset \mathrm{Sym}(l^\vee) \otimes_A M$$

Where $c_{|l}$ is the element in degree 1 corresponding to the inclusion $l \subset M$ via the natural isomorphism $l^\vee \otimes_A M \to \mathrm{Hom}_A(l, M)$. Applying the $\widetilde{(-)}$ here correspond to picking a generator of $l^\vee$, say the restriction of $\varphi \in M^\vee$ with $\varphi(l) \neq 0$ and applying the homogeneous localization at φ. Recall that as in the above exercise we have

$$\mathrm{Sym}(l^\vee)_{(\varphi)} = k.$$

Say v is a generator of l such that $\varphi(v) = 1$. Then, $x_{i|l} = x_i(v)\varphi_{|l}$ in $l^\vee$. Now the homogeneous localization of $c_{|l}$ is

$$\sum_i \frac{x_{i|l}}{\varphi_{|l}} \otimes e_i = \sum_i x_i(v) \otimes e_i \to \sum_i x_i(v) e_i = v,$$

along the natural isomorphism $k \otimes_k M \to M$ and using Equation (1). We finally get that the k-linear subspace generated by the homogeneous localization of $c_{|l}$ in M is l. In other words, the fiber of $\mathcal{O}(-1) \subset \pi^* M$ at the point $l \subset M$ of $\mathbb{P}(M)$ is

$$l \subset M.$$

This is why $\mathcal{O}(-1)$ is called the *tautological line bundle.*

□

Remark. The definition of c is invariant under base change and is invariant under isomorphisms of M. What we mean by this is that for any morphism $A \to A'$ we can base change all the above setup to A' and all the definitions commute to base change. Namely there is a natural isomorphism

$$(\mathrm{Sym}(M^\vee) \otimes_A M) \otimes_A A' \to \mathrm{Sym}(M'^\vee) \otimes_A' M'$$

where $M' = M \otimes_A A'$ and that under this natural isomorphism c is sent to c' where c' would be defined exactly as c but for M'. Also, if $\psi\colon M \to N$ is an isomorphism, note that we have the following square of commuting isomorphisms of A-modules

$$\begin{array}{ccc} M^\vee \otimes_A M & \longrightarrow & \mathrm{Hom}_A(M,M) \\ {\scriptstyle \psi^{-1\vee}\otimes\psi}\downarrow & & \downarrow{\scriptstyle \psi\circ - \circ\psi^{-1}} \\ N^\vee \otimes_A N & \longrightarrow & \mathrm{Hom}_A(N,N) \end{array}$$

which shows that $\psi^{-1\vee} \otimes \psi$ is an isomorphism that preserves c. Using the two fact above, we can harmlessly reduce to the case where M is free for the above proof: one first localize on A to have that M is isomorphic to a free module, and the use a fixed isomorphism to really make in sort that M is equal to a free module using the above square.

Exercise 5. *Euler sequence.* Let A be a ring and M a finite projective A-module.

(1) *Directional derivative.* For a $v \in M$, show that there is a unique A-derivation
$$\frac{\partial}{\partial v}\colon \mathrm{Sym}(M^\vee) \to \mathrm{Sym}(M^\vee)$$
which is equal to the evaluation at v on elements of degree 1. If M is free, if (e_i) and (x_i) denotes a basis and a dual basis respectively, and $v = \sum \lambda_i e_i$, show that
$$\frac{\partial}{\partial v} = \sum_i \lambda_i \frac{\partial}{\partial x_i}.$$

(2) For $\varphi \in M^\vee$, show that $\frac{\partial}{\partial v}$ uniquely extends to an A-derivation
$$\frac{\partial}{\partial v}\colon \mathrm{Sym}(M^\vee)_\varphi \to \mathrm{Sym}(M^\vee)_\varphi.$$
Deduce that $\frac{\partial}{\partial v}$ defines an A-derivation[6],
$$\frac{\partial}{\partial v}\colon \mathrm{Sym}(M^\vee)_{(\varphi)} \to \mathrm{Sym}(M^\vee)(-1)_{(\varphi)}.$$

(3) Denote by $\pi\colon \mathbb{P}(M) \to \mathrm{Spec}(A)$ and $\mathcal{T}^1_{\mathbb{P}(M)|A} = \left(\Omega^1_{\mathbb{P}(M)|A}\right)^\vee$. Deduce from the above that there is a $\mathcal{O}_{\mathbb{P}(M)}$-linear map
$$\frac{\partial}{\partial(-)}\colon \pi^* M \to \mathcal{T}^1_{\mathbb{P}(M)|A}(-1).$$
Hint: $\mathcal{T}^1_{\mathbb{P}(M)|A}(-1) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}(M)}}(\Omega^1_{\mathbb{P}(M)|A}, \mathcal{O}(-1))$[7]*. Use the universal property of* $\Omega^1_{\mathbb{P}(M)|A}$ *on affines* $D_+(\varphi)$.

[6] where $(-)(-1)$ denotes tensoring by $\mathcal{O}(-1)$.

[7] Because in general if $\mathcal{F}$ is finite locally free and $\mathcal{G}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{F}^\vee \otimes \mathcal{G} \cong \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$

(4) *Euler sequence.* Show that there is an exact sequence of $\mathcal{O}_{\mathbb{P}(M)}$-locally free sheaves

$$0 \to \mathcal{O}(-1) \to \pi^* M \xrightarrow{\frac{\partial}{\partial(-)}} \mathcal{T}^1_{\mathbb{P}(M)|A}(-1) \to 0$$

where the first arrow is the canonical inclusion $\mathcal{O}(-1) \to \pi^* M$ and the second is the arrow above. *Hint: show the sequence is exact when restricted to standard opens.*

Solution key. (1) First we consider the following situation. Say P is an A-module. An A-linear derivation

$$\mathrm{Sym}_A(P) \to N$$

for a $\mathrm{Sym}_A(P)$-module N is a A-linear map satisfying the Leibniz rule. This is the same as a map $\mathrm{Sym}_A(P)$-algebra map

$$\mathrm{Sym}_A(P) \to \mathrm{Sym}_A(P) \oplus_0 N$$

which is a section of the projection. By the universal property of the symmetric algebra, we deduce that this is same as the data of an A-linear map

$$P \to \mathrm{Sym}_A(P) \oplus_0 N$$

such that the first component is the natural degree 1 inclusion $P \to \mathrm{Sym}_A(P)$. Therefore we deduce that this data is equivalent to an A-linear map

$$P \to N$$

All in all we have shown that there is a natural isomorphism in N

$$\mathrm{Der}_A(\mathrm{Sym}_A(P), N) \cong \mathrm{Hom}_A(P, N) \cong \mathrm{Hom}_{\mathrm{Sym}_A(P)}(\mathrm{Sym}_A(P) \otimes_A P, N).$$

We therefore deduce that

$$\Omega_{\mathrm{Sym}_A(P)|A} = \mathrm{Sym}_A(P) \otimes_A P.$$

The universal derivation $\mathrm{Sym}_A(P) \to \mathrm{Sym}_A(P) \otimes_A P$ being entirely determined by sending $p \in P$ to $1 \otimes p$ and then defined by the Leibniz rule.

In particular for $P = M^\vee$, for any $v \in M$ there is a unique derivation corresponding to the A-linear map

$$\mathrm{ev}_m \colon M^\vee \to \mathrm{Sym}(M^\vee).$$

(2) Follows from the universal property of the derivations and because derivating with respect to a variable of degree 1 decrease the degree of polynomnial by 1, as usually implied by the Leibniz rule.

(3) On an affine $D_+(\varphi)$ we define

$$M \otimes \mathrm{Sym}(M^\vee)_{(\varphi)} \to \mathrm{Der}_A(\mathrm{Sym}(M^\vee)_{(\varphi)}, \mathrm{Sym}(M^\vee)_{(\varphi)}(-1))$$

by sending $v \otimes \frac{f}{\varphi^{\deg(f)}}$ to the derivation $\frac{f}{\varphi^{\deg(f)}} \frac{\partial}{\partial v}$. This glues because this only depends on where to send M.

(4) Working locally is enough. Indeed the sequence is functorial in $\mathcal{E}$, so this is sufficient. Let's say that $\mathcal{E} = A^{n+1}$. Fix an $i \in \{0, \dots, n\}$, without loss of generality say $i = 0$. Denote by $t_j = \frac{x_j}{x_0}$. Note that (we just need to choose where to send each t_j)

$$\mathrm{Der}_A(A[t_1, \dots, t_n], \frac{1}{x_0} A[t_1, \dots, t_n]) = \bigoplus_{j \geq 1} A[t_1, \dots, t_n] \frac{1}{x_0}.$$

In term of the above basis, we have $\frac{\partial}{\partial x_0} = (-\frac{x_1}{x_0}, \dots, -\frac{x_n}{x_0})$, because

$$\frac{\partial}{\partial x_0}(\frac{x_j}{x_0}) = \frac{-x_j}{x_0^2} = \frac{1}{x_0}\frac{-x_j}{x_0}.$$

Also we have $\frac{\partial}{\partial x_j} = e_j$. As $e_i \in (A[t_0, \dots, t_n])^{\oplus n+1}$ is sent to $\frac{\partial}{\partial x_i}$ we get that the martix of the map is

$$\begin{pmatrix} -\frac{x_1}{x_0} & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{x_n}{x_0} & 0 & \cdots & 1 \end{pmatrix}.$$

It follows that the map is surjective and that the kernel is generated by

$$\sum_{j=0}^{n} e_j \otimes \frac{x_j}{x_0} = \frac{c}{x_0}$$

which proves the claim. (We used the notation c from the exercise on the tautological line bundle).

□

Note the following lemmas about inclusion of vector bundles.

Lemma. Let $\mathcal{E} \to \mathcal{F}$ be an inclusion of finite locally free sheaves on a scheme X. Then the following are equivalent.

(1) For every $x \in X$ the fiber $\mathcal{E}(x) \to \mathcal{G}(x)$ is injective.
(2) The dual map $\mathcal{F}^\vee \to \mathcal{E}^\vee$ is surjective.
(3) The cokernel $\mathcal{F}/\mathcal{E}$ is locally free.

Proof. Because the statement is local and invariant under isomorphism, we are in the situation of an injective map

$$R^{\oplus n} \to R^{\oplus m}$$

between free modules for some ring R, given by some matrix M.

We show that (1) implies (2). The fiber of $M(\mathfrak{p})^T$ of the transpose of the matrix is injective because the dual map correspond to the transpose matrix, so we just use the hypothesis (1). Now it implies that $M(\mathfrak{p})$ defines a surjective map because this is true for vector spaces that the dual of an injective map is surjective.[8]Therefore by Nakayama we deduce that the dual map is surjective, proving (2).

[8]Which is false for general modules! Look at $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$.

Now we prove that (2) implies (3). Say that we are given an injective map $N \to M$ between finite projective modules. Therefore we have a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M/N, R) \longrightarrow \mathrm{Hom}_R(M, R) \longrightarrow \mathrm{Hom}_R(N, R) \dashrightarrow 0$$

where the dashed arrow is the hypothesis (2). Because the kernel of a map between finite projective modules is finite projective, then $\mathrm{Hom}_R(M/N, R)$ is projective. Now, taking the dual of an exact sequence of finite projective modules is again a short exact sequence of finite projective modules, so we have a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \dashrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N^{\vee\vee} & \longrightarrow & M^{\vee\vee} & \longrightarrow & (M/N)^{\vee\vee} & \longrightarrow & 0 \end{array}$$

so we conclude using the five lemma that $M/N \cong (M/N)^{\vee\vee}$. But $(M/N)^{\vee\vee}$ is the dual of $(M/N)^{\vee}$ which was shown above to be locally free, so we can conclude because the dual of a locally free sheaf is locally free.

We conclude by showing that (3) implies (1). Namely we use the exact sequence

$$0 = \mathrm{Tor}_1^R(k(\mathfrak{p}), M/N) \longrightarrow N(\mathfrak{p}) \longrightarrow M(\mathfrak{p}) \longrightarrow (M/N)(\mathfrak{p}) \longrightarrow 0$$

which concludes. The first group is zero because M/N is flat. $\square$

We say that $N \to M$ is a *sub-bundle* when it satisfies one of the equivalent condition of the lemma.