

Solutions – week 8

Remark. *The Jacobian criterion for hypersurfaces.* In the following, we explain the so called *Jacobian criterion*, in a simple case. Let k be a field. Consider $f \in k[x_1, \dots, x_n]$ a non-zero polynomial. Let

$$R = k[x_1, \dots, x_n]/f.$$

Denote also by $I = (f)$. Note that the unique map of $k[x_1, \dots, x_n]$ -modules $k[x_1, \dots, x_n] \rightarrow I$ sending 1 to f is an isomorphism, so that I is a free $k[x_1, \dots, x_n]$ -module of rank 1. Therefore we have a map of free R -modules

$$(1) \quad R \otimes_{k[x_1, \dots, x_n]} I = I/I^2 \rightarrow \bigoplus_{i=1}^n R dx_i$$

sending the generator f to the exterior derivative

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \in \bigoplus_{i=1}^n R dx_i.$$

Then R is smooth over k if and only if the map from Equation (1) is injective with a projective cokernel. Indeed: the cokernel is $\Omega_{R|k}$. Linear algebra translates this condition to: for every $\mathfrak{p} \in \text{Spec}(R)$ the fiber at \mathfrak{p} of the the map in Equation (1) is not zero. This is equivalent to saying that the closed subscheme $V(\frac{\partial f}{\partial x_i})_{i=1}^n$ in $\text{Spec}(R)$ is empty. Indeed points of this closed subscheme are precisely points where $\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i = 0$. In the end we can reformulate that R is smooth over k if and only if the following closed subscheme of \mathbb{A}_k^n is empty:

$$V\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, f\right) = \emptyset.$$

Exercise 1. *Jacobian criterion.* Let k be an algebraically closed field. Are the following schemes smooth over k ?

- (1) $V_+(XZ - Y^2) \subseteq \mathbb{P}_k^2$
- (2) $V(xz - y^2) \subseteq \mathbb{A}^3$
- (3) $V_+(XZ - Y^2), V_+(YW - Z^2), V_+(XW - YZ) \subseteq \mathbb{P}_k^3$
- (4) $V(y^2 - x(x-1)(x+1)) \subseteq \mathbb{A}_k^2$

Hint: Be careful about the characteristic of k !

Exercise 2. *Regularity vs smoothness.* Let $k = \mathbb{F}_p(t)$, and consider $X = V(x^p - t) \subseteq \mathbb{A}_k^1$. Show that X is regular.¹

¹Recall that a Noetherian local ring (R, \mathfrak{m}) is regular if and only if $\dim(R) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$. Do you see why this first question is silly?

However, let $k' = \mathbb{F}_p(t^{1/p}) \supseteq k$. Then show that the base change $X_{k'}$ is not regular.

Was X smooth over k in the first place?

Exercise 3. *Separable extensions and differentials.* Let k be a field and l a finite extension. Show that $\Omega_{l|k}^1 = 0$ if and only if l is a separable extension.

Solution key. If l is separable and finite then $l = k(\alpha)$ for some algebraic and separable element α . But then $l = k[t]/(f(t))$ and $f'(t)$ is not zero in the quotient by separability. This concludes one way by the conormal sequence. For the other direction, if l is not separable, then there is some $\alpha \in l$ such that $l'(\alpha) = l$ for some sub-extension l' such that the extension is not separable implying that $l = l'[t]/(f(t))$ with the derivative vanishing in this quotient. Therefore $\Omega_{l|l'} = l$ by the conormal sequence. But if $\Omega_{l|k} = 0$ then $\Omega_{l|l'} = 0$ by the fundamental sequence of cotangent sheaves, which is a contradiction. □

Exercise 4. *Relative Spec.* Let S be a scheme. Let \mathcal{A} be a quasi-coherent \mathcal{O}_S -algebra. This means that it is a sheaf \mathcal{O}_S -algebras which is quasi-coherent as an \mathcal{O}_S -module.

- (1) Let $V \subset U \subset S$ two open affines. Show that the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{A}(V)) & \longrightarrow & \mathrm{Spec}(\mathcal{A}(U)) \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \end{array}$$

is cartesian.

- (2) Let $X = \bigcup U_i$ be an affine cover. Deduce that we can glue the schemes $(\mathrm{Spec}(\mathcal{A}(U_i)))$ to an S -scheme

$$\underline{\mathrm{Spec}}_S(\mathcal{A}) \rightarrow S.$$

- (3) Show that $\underline{\mathrm{Spec}}_S(\mathcal{A})$ satisfies the following universal property in the category of S -schemes. If $f: T \rightarrow S$ is an S -scheme then a S -morphism $T \rightarrow \underline{\mathrm{Spec}}_S(\mathcal{A})$ is the same as a morphism of \mathcal{O}_T -algebras $f^*\mathcal{A} \rightarrow \mathcal{O}_T$. Deduce that $\underline{\mathrm{Spec}}_S(\mathcal{A})$ is independent of the affine cover for the construction.
- (4) Let $f: X \rightarrow Y$ be an affine morphism of schemes. Show that there is a natural isomorphism of Y -schemes $X \cong \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$.
- (5) Let \mathcal{E} be a locally free sheaf of finite rank on S . We define the *schematic vector bundle associated to \mathcal{E}* by

$$\mathbb{V}(\mathcal{E}) = \underline{\mathrm{Spec}}_S(\mathrm{Sym}(\mathcal{E}^\vee))$$

where the \mathcal{O}_S -algebra $\text{Sym}(\mathcal{E}^\vee)$ denotes the free \mathcal{O}_S -algebra generated by \mathcal{E}^\vee .² Show that a S -morphism from $f: T \rightarrow S$ to $\mathbb{V}(\mathcal{E})$ is the same as a global section of $f^*(\mathcal{E})$, i.e. an element of $f^*(\mathcal{E})(T)$.

- (6) Show that there is always a canonical section of $p: \mathbb{V}(\mathcal{E}) \rightarrow S$ which correspond to $0 \in \mathcal{E}(S)$ which defines a closed subscheme of $\mathbb{V}(\mathcal{E})$ isomorphic to S . We call this closed subscheme the *zero section* of $\mathbb{V}(\mathcal{E})$.

Solution key. (1) We may cover V by principal opens affine to prove the isomorphism locally. In this case it is clear that it follows from quasi-coherence.

- (2) Using the above we see that $\text{Spec}(\mathcal{A}(U_{ij})) \rightarrow \text{Spec}(\mathcal{A}(U_i))$ are open immersions, being locally pullbacks of open immersions. It follows that we can glue this to a scheme.
- (3) Cover S by affine schemes U_i . It induces an open cover of T by open subschemes T_i . Suppose we are given $f: T \rightarrow \underline{\text{Spec}}_S(\mathcal{A})$. Note that this corresponds by gluing to a collection of maps of U_i -schemes $f: T_i \rightarrow \text{Spec}(\mathcal{A}(U_i))$ that appropriately glues. This correspond one to one to a collection of $\mathcal{O}(U_i)$ -algebra maps $\mathcal{A}(U_i) \rightarrow \mathcal{O}(T_i)$ which correspond one to one to morphisms of \mathcal{O}_S -algebras $\mathcal{A} \rightarrow f_*\mathcal{O}_T$, from which the claim follows by adjunction.
- (4) As f is affine, note that $f_*\mathcal{O}_X$ is quasi-coherent. Note also that for an open affine $U \subset Y$, we have a natural identification $f^{-1}(U) = \text{Spec}(\mathcal{O}(f^{-1}(U))) = \text{Spec}(f_*\mathcal{O}_X(U))$ from which the claim follows.
- (5) By the above, such a morphism is the same as the data of an \mathcal{O}_T -algebra morphism $\text{Sym}(f^*\mathcal{E}^\vee) \rightarrow \mathcal{O}_T$. Therefore this the same as an \mathcal{O}_T -module morphism $f^*\mathcal{E}^\vee \rightarrow \mathcal{O}_T$. By duality, this is the same as a section of $f^*\mathcal{E}$.
- (6) Affine locally on S , say on some affine $\text{Spec}(R)$, we can assume that $\mathcal{E} \cong \widetilde{R^n}$ is finite free, and the zero section correspond to the origin of \mathbb{A}_R^n .

□

²It's a gluing of the usual construction in linear algebra.