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AG II - Schemes and sheaves

Solutions – week 7

Exercise 1. *Regular proper curves are projective.* Let k be a field. In this exercise, we say that a 1-dimensional finite type separated normal scheme over k is a *regular curve over* k. Recall that a local Noetherian normal ring is a DVR.[1] We say that a *projective* k*-scheme* is a closed sub-scheme of $\mathbb{P}^n_k$ for some $n \geq 1$.

(1) *Extension principle.* Let C be regular curve over k. Let $x \in C$ be a closed point and $U = C \setminus \{x\}$. Let $Y \to \operatorname{Spec}(k)$ be proper. Show, using the valuative criterion for properness, that any map $U \to Y$ of k-schemes extend uniquely to a map $C \to Y$.

(2) Show that any proper regular curve C is a k-projective scheme. You can proceed as follows.

(a) Let $(U_i)_{i=1}^n$ be an open cover of C by affine (regular) curves. Show that for each i there is a morphism of k-schemes $U_i \to Y_i$ which is an open immersion where Y_i is a k-projective scheme which is integral, separated and finite type.

(b) *The product of projective schemes is projective.* Let

$$Y = Y_1 \times_k \cdots \times_k Y_n.$$

Show that Y is a k-projective scheme.

(c) Using the *extension principle* show that the natural map

$$\bigcap_{i=1}^{n} U_i \to Y$$

extends uniquely to a map $\varphi\colon C \to Y$.

(d) Using Exercise 4, consider $Z := \varphi(C) \subset Y$ the closed subscheme with the reduced structure of Y. Show that the induced map $\psi\colon C \to Z$ is an isomorphism and conclude.

Hint: Using that the map $C \to Z$ is surjective by construction, and that for every i

$$U_i \to Z \to Y_i$$

is an open immersion, show that for every $c \in C$ we have that the map $\mathcal{O}_{Z,f(c)} \to \mathcal{O}_{C,c}$ is an isomorphism. Conclude that Z is a regular curve and that for any i, $U_i \to Z$ is an open immersion. Therefore extend the inclusion $U_i \to C$ map to a map $Z \to C$ and show that this will be an inverse map to the map constructed above.

[1] See Tag 00PD for example.

Solution key. (1) Note that we have by the valuative criterion an unique extension

$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec}(\mathcal{O}_{C,x}) & \longrightarrow & \operatorname{Spec}(k) \end{array}$$

Say that $V \subset Y$ is an open affine which is an open neighborhood of the image of the closed point in $\operatorname{Spec}(\mathcal{O}_{C,x})$. Because the image of the open point is a generalization of the image the closed point we see that the we have a factorization $\operatorname{Spec}(\mathcal{O}_{C,x}) \to V$.

We have that $\mathcal{O}(V)$ is a finite type k-algebra. Consider the induced map $\mathcal{O}(V) \to \mathcal{O}_{C,x}$. Generators are sent to elements of the form $\frac{f}{g}$ where $f, g \in \mathcal{O}(U)$ for some open affine $U \ni x$ in C. By localizing at a finite number of denominators we can suppose that the image of the generators are in $\mathcal{O}(U)$ for a potentially smaller open affine U. Therefore we get a map $U \to V \to Y$. Now $U \to Y$ and $C \setminus \{x\} \to Y$ necessarily glues because they agree on the generic point and by Exercise 5.

(2) (a) Say that $U = \operatorname{Spec}(A)$ is the spectrum of a finite type k-algebra which is a domain. We may write it as

$$\frac{k[x_1/x_0, \ldots, x_n/x_0]}{\mathfrak{p}}$$

where $\mathfrak{p}$ is a prime ideal. Define the homogeneous ideal

$$\mathfrak{p}_+ = (\{g \in k[x_0, \ldots, x_n] \mid g \text{ is homogeneous and } \quad x_0^{-\deg(g)} g \in \mathfrak{p}\}).$$

As $\mathfrak{p}$ is prime, and that we can check that an homogeneous ideal is prime by on homogeneous elements, we see that $\mathfrak{p}_+$ is also prime. Now, by construction

$$\operatorname{Spec}(A) \cong D_+(x_0) \subset \operatorname{Proj}\left(\frac{k[x_0, \ldots, x_n]}{\mathfrak{p}_+}\right).$$

Concluding.

(b) By induction n, it suffices to show that the product of two projective spaces

$$\mathbb{P}^n_k \times_k \mathbb{P}^m_k$$

is a k-projective scheme. Write

$$\mathbb{P}^n_k = \operatorname{Proj}(k[x_0, \ldots, x_n]) \quad \mathbb{P}^m_k = \operatorname{Proj}(k[y_0, \ldots, y_m])$$

Then we claim that

$$\operatorname{Proj}(k[x_i y_j]_{i,j}) \cong \mathbb{P}^n_k \times_k \mathbb{P}^m_k,$$

which would conclude because $k[x_i y_j]_{i,j}$ is a finitely generated k-algebra.[2] Namely fix k and l, and see that

$$(k[x_i y_j]_{i,j})_{(x_k y_l)} = k[\frac{x_i}{x_k}] \otimes_k k[\frac{y_j}{y_l}]$$

[2]Be careful these generators have relations, namely $(x_i y_j)(x_k y_l) = (x_i y_l)(x_k y_j)$. One can show that these are the only relations.

which are the functions of $D_+(x_k) \times_k D_+(y_l)$. These identifications glue as desired.

(d) Define

$$\mathcal{I} = \ker(\mathcal{O}_Y \to \varphi_* \mathcal{O}_C).$$

Because φ is quasi-compact $\varphi_* \mathcal{O}_C$ is quasi-coherent. Therefore $\mathcal{I}$ is quasi-coherent ideal because $\mathrm{Qcoh}(Y)$ is an abelian category. We claim that $V(\mathcal{I}) = \varphi(C)$ topologically and that $V(\mathcal{I})$ is reduced as a scheme. Reducedness follows from the fact C is reduced so that $\mathcal{O}(C)(U)$ is a reduced ring for any open U. Let $Z' = V(\mathcal{J})$ be another closed subscheme such that we have a factorization

$$\begin{array}{ccc} C & \dashrightarrow & Z' \\ & \searrow^{\varphi} & \downarrow \\ & & Y \end{array}$$

Then we have maps

$$\begin{array}{ccc} \varphi_* \mathcal{O}_C & \longleftarrow & \mathcal{O}_Y / \mathcal{J} \\ & \nwarrow & \uparrow \\ & & \mathcal{O}_Y \end{array}$$

which implies that we have $\mathcal{O}_Y/\mathcal{J} \to \mathcal{O}_Y/\mathcal{I}$ and therefore that $\mathcal{J} \subset \mathcal{I}$ and that $V(\mathcal{I}) \subset V(\mathcal{J})$. The above reasoning shows that $\overline{\varphi(C)} = V(\mathcal{I})$ topologically (seeing the closure as the intersections of all closed-subsets containing the image). But because $C \to Y$ is a k-map between proper schemes over k, the image is closed, concluding.

Now by construction we have a scheme theoretic map

$$C \to V(\mathcal{I}) \subset Y$$

where $V(\mathcal{I})$ is $\varphi(C)$ with the reduced structure. Let $Z := V(\mathcal{I})$. We now show that $C \to Z$ is an isomorphism. Note that $U_i \to Z \to Y_i$ is by construction an open immersion. Say $c \in U_i$. The we have maps

$$\mathcal{O}_{Y_i, \varphi_i(c)} \to \mathcal{O}_{Z, \varphi(c)} \to \mathcal{O}_{C,c}$$

such that the composition is an isomorphism. Note that because Z and C are reduced and that $C \to Z$ is surjective, we have that $\mathcal{O}_{Z,\varphi(c)} \to \mathcal{O}_{C,c}$ is injective, and therefore an isomorphism. It implies that there is an isomorphism at residue fields. Therefore the image of a closed point is closed and the image of the generic point is also the unique generic point of Z. It follows that Z is a regular curve.

Because φ is injective when restricted to any U_i, we see that $\varphi \colon U_i \to Z$ is an open immersion. Indeed: let $c \in U_i$ and V be an open affine neighborhood of $\varphi(c)$. Then $\mathcal{O}(V)$ is a finite type k-algebra. Therefore when looking at

$$\mathcal{O}_Z(V) \to \mathcal{O}_{Z,\varphi(c)} \to \mathcal{O}_{C,c}$$

this map uniquely lifts to map

$$\mathcal{O}_Z(V) \to \mathcal{O}_C(W)$$

where $W \subset U_i$ is some affine open. This is done as in point (1) of the exercise. Now, this map of $\mathcal{O}_Z(V)$-modules is an isomorphism at every prime localization, implying that this map is an isomorphism. Now, because $\varphi\colon U_i \to Z \to Y_i$ is injective, the map $U_i \to Z$ is injective and a local isomorphism of schemes, *i.e.* an open immersion of schemes.
Now, we are able to extend for some i the map $U_i \to C$ to a map $Z \to C$ using the extension principle. Using uniqueness of extensions, this map is necessarily an inverse to the map constructed above.

□

Exercise 2. *An open of an affine is not necessarily affine.* Let R be a non-zero ring. Show that $U = \mathrm{Spec}(R[x,y]) \setminus V(x,y)$ is not affine.
Hint: compute $\mathcal{O}(U)$ using an appropriate cover and the sheaf property.

Solution key. We use the cover $D(x) \cup D(y)$ and the sheaf property to compute global sections of U. Because x, y are non zero divisors, localization maps $R[x^{\pm 1}, y] \to R[x^{\pm 1}, y^{\pm 1}]$ and $R[x, y^{\pm 1}] \to R[x^{\pm 1}, y^{\pm 1}]$ are injective and we may treat them as inclusions. Now, global sections are the elements of the kernel of the map

$$R[x^{\pm 1}, y] \times R[x, y^{\pm 1}] \to R[x^{\pm 1}, y^{\pm 1}]$$

that sends $(f, g) \mapsto f - g$. In other words

$$\mathcal{O}(U) = R[x^{\pm 1}, y] \cap R[x, y^{\pm 1}] = R[x, y].$$

Indeed if $fy^n = gx^m$ for some n, m, then $f = 0 \mod x^m$. But y is a non-zero divisor in $R[x,y]/(x^m)$. Therefore $f \in (x^m)$. A symmetric argument concludes the claim equality.
If U was affine, then the natural $U \to \mathrm{Spec}(R[x,y])$ would be an isomorphism, because it an inclusion of an open, an equality. But because $R \neq 0$, $\mathrm{Spec}(R[x,y] \setminus U) = \mathrm{Spec}(R)$ is non empty, a contradiction.

□

Exercise 3. *Intersection of affine schemes.* Let X be a scheme and $U, V \subset X$ be open affine sub-schemes.

(1) Show that if X is separated then $U \cap V$ is affine.
Hint: Show that $U \cap V \cong X \times_{X \times X} (U \times V)$.
(2) Show that $U \cap V$ is not necessarily affine if X is not separated.
Hint: remember this open of an affine which is not affine? Play with this.

Solution key. For the first point, the claim follows from the Hint because the intersection is realized has a closed subscheme of an affine scheme. For the second point, one can take the affine plane with two origins. $\square$

Exercise 4. *A map from a proper scheme to a separated scheme is closed.* Let $f\colon X \to Y$ be a map of S-schemes. Suppose that $Y \to S$ is separated.

(1) Show that the graph $(\mathrm{id}, f) = \Gamma_f\colon X \to X \times_S Y$ is a closed immersion.
(2) Let $Z \subset X$ a closed subscheme proper over S. Show that $f_{|Z}$ is closed.

Remark. This fact is analogue to the topological result that a continuous map from a compact topological space to a Hausdorff space is always closed.

Solution key. The first point follows because

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \Gamma_f}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times_S Y & \xrightarrow{f \times \mathrm{id}} & Y \times_S Y \end{array}$$

is a pullback square. The second claim follows because $Z \to Z \times_S Y \to Y$ is closed, the first map being closed by the first point and the second map being closed by universal closedness of $Z \to S$.

$\square$

Exercise 5. *Morphisms into separated schemes.* Let S be a scheme. Let $X \to S$ and $Y \to S$ be S-schemes. Suppose that X is reduced and $Y \to S$ separated. Show that two morphisms of S-schemes

$$f_1, f_2\colon X \to Y$$

that coincide an open dense subset of X are equal.
Give counter-examples if one of the hypotheses is dropped.

Remark. This fact is analogue to the topological result that if two continuous morphisms to a Hausdorff space agree on a open dense then they actually agree everywhere.

Solution key. Let Z be the scheme where $f_1 = f_2$ *i.e.* the pullback

$$\begin{array}{ccc} Z & \xrightarrow{f_1 = f_2} & Y \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow{f_1 \times f_2} & Y \times_S Y \end{array}$$

Because Y is separated Z is closed in X. Because of the assumption, $Z = X$ topologically, but then schematically because X is reduced.

We provide a counter-example if X is not reduced. Consider the two k-algebras maps (k being a field say)

$$k[x] \mapsto k[x, y]/(xy, y^2)$$

sending x to x and $x + y$ respectively. The induced maps on Spec agree on $D(x)$ which is dense. □