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AG II - Schemes and sheaves

Solutions – week 6

See the *blow-up document on Moodle for detailed solutions of exercise 1 and 2.*

Exercise 1. *Blow-ups.* Let R be a ring and $I \subset R$ an ideal. We define the blow-up of $\mathrm{Spec}(R)$ at $V(I)$ to be the map ($I^0 = R$)

$$b\colon \mathrm{Bl}_I = \mathrm{Proj}(\bigoplus_{n\geq 0} I^n) \to \mathrm{Spec}(R).$$

The *exceptional divisor* of the blow-up is the closed subscheme of $\mathrm{Proj}(\bigoplus_{n\geq 0} I^n)$

$$E = V_+(\bigoplus_{n\geq 0} I^{n+1}).$$

(1) Show that b defines an isomorphism of schemes

$$b\colon \mathrm{Bl}_I \setminus E \to \mathrm{Spec}(R) \setminus V(I).$$

(2) Let A be a ring, $R = A[x_0, \dots, x_n]$ and $I = (x_0, \dots, x_n)$. Show that $E \cong \mathbb{P}^n_A$.
Remark. Let us introduce a bit of intuition. Points (1) and (2) subsume the key philosophy of blow-ups. First of all a blow-up is a map which is an isomorphism outside of a the fiber closed subscheme $V(I)$. We will see later that in nice cases I/I^2 is to be interpreted as the *conormal bundle* of $V(I)$ in $\mathrm{Spec}(R)$ (*i.e.* tangent vectors going out of $V(I)$). Therefore E can be interpreted as the projective space of the vector space of directions outside of $V(I)$. Meaning that for each direction going outside of $V(I)$, there is a corresponding point in E. For example, in the actual computation above, the exceptional divisor E is the space of lines through the origin in $\mathbb{A}^{n+1}$, i.e. $\mathbb{P}^n$.

(3) *Standard blow-up charts.* Consider the same setting as in the last item. Show that Bl_I can be identified as the scheme

$$V_+(x_iY_j - x_jY_i)$$

inside $\mathbb{A}^{n+1}_A \times \mathbb{P}^n_A = \mathrm{Proj}(A[x_0, \dots, x_n] \otimes_A A[Y_0, \dots, Y_n])$ where the grading is taken to be the Y-grading.

Remark. See here for a representation of the blow-up at (x_0, x_1) of $\mathbb{A}^2$ using the standard charts. The projection to the x, y-plane is a bijection outside of the pre-image of the origin which is a line.

Exercise 2. *Strict transforms.* Let R be a ring. Let I be an ideal and $b\colon \mathrm{Bl}_I \to \mathrm{Spec}(R)$ the blow-up at the ideal I. Let $J \subset R$ be another ideal. We define the *strict transform of* $V(J)$ to be the blow-up of $V((I+J)/J)$ in $\mathrm{Spec}(R/J)$.

(1) Show that St_J can be identified with be the closed subscheme of Bl_I

$$V_+\left(\bigoplus_n I^n \cap J\right).$$

(2) Show that b induces an isomorphism

$$b\colon \mathrm{St}_J \setminus E \to V(J) \setminus V(I).$$

(3) *Resolving a singularity.* Let k be a field. Compute the strict transform with $R = k[x_0, x_1]$, the ideal $I = (x_0, x_1)$ and $J = (x_1^2 - (x_0^3 + x_0^2))$. Use the standard blow-up charts. Show that this strict transform is regular.

Remarks. The equation of the last item is the equation of a *nodal curve* which is a type of singular curve. See here for a representation. The tangent space at the origin has dimension 2, which is why the curve is not regular. Since the blow-up at a point replaces a point by "directions out of the point", it is no surprise that blowing up a node at its nodal point removes the singularity.

Exercise 3. *Examples and computations of blow-ups.* Let k be an algebraically closed field. You can use the following.
Let $A = k[x_1, \dots, x_n]/(f)$. Denote by $\partial_i f$ the derivative of f with respect to x_i. Then

$$\mathrm{Spec}(A) \text{ is regular} \iff V(f, \partial_1 f, \cdots, \partial_n f) = \emptyset.$$

Moreover $V(f, \partial_1 f, \cdots, \partial_n f)$ consists exactly of the non-regular points of $\mathrm{Spec}(A)$.

(0) Let R be a ring. Show that if $I = (f_0, \dots, f_n)$ is generated by a regular sequence then $\mathrm{Bl}_I = V_+(X_i f_j - X_j f_i)$ in $\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times \mathrm{Spec}(R)$. (Use the lemmas in the blow-ups document from moodle)
(1) Show that blow-up of (x^2, y) in $\mathrm{Spec}(k[x, y])$ is not regular. What are the regular points?[1]
(2) Show that $X = \mathrm{Spec}(k[x, y, z, w]/(xy - zw))$ is not regular. What are the regular points?
(3) Show that blow-ups of X at (x, y, z, w) and (x, z) are regular. We denote these blow-ups by $X_1 \to X$ and $X_2 \to X$.
Remark. This is another example where blow-ups resolves (=removes) singularities.
(4) Compute fibers of (x, y, z, w) of $X_1 \to X$ and $X_2 \to X$.

Solution key. This was an hand-in exercise a previous year so solutions are credited to students who wrote them.
(1)(Joel) Let $A = k[x, y]$, $I = (x^2, y^2)$ and $R = A/I$. Consider the map $\phi : R[Z, W] \twoheadrightarrow \bigoplus_{n \geq 0} I^n$ which sends $Z \to x^2$ and $W \to y^2$ in degree one. Then $\ker \phi = (Zy - Wx)$, so $\mathrm{Bl} \cong \mathrm{Proj}\ R[Z, W]/(Zy^2 - Wx^2)$. Next, we

[1] This investigation can be used to show that this blow-up is normal.

show that the blow-up is not normal. Consider the affine chart U_W where $W \neq 0$, which is given by $\operatorname{Spec} k[x,y,z]/(zy^2 - x^2) =: \operatorname{Spec} A$, where $z = \frac{Z}{W}$. Then $\frac{zy}{x} \in \operatorname{Frac}(A)$, and $\left(\frac{zy}{x}\right)^2 = \frac{z \cdot zy^2}{x^2} = \frac{zx^2}{x^2} = z$. Hence, $\frac{zy}{x}$ is a root of the monic polynomial $P(t) = t - z$ with coefficients in A. Now $\frac{zy}{x} \notin A_{((x,y))}$, as x is not inverted in the localization, and the field of fractions of $A_{((x,y))}$ is the same as for A, we see that the blow-up is not normal, therefore, not regular.

(2)(Julie) Let $g = xy - zw \in k[x,y,z,w]$. By the criterion provided in the statement of the exercise, the set of non-regular points in

$$\operatorname{Spec}\left(\frac{k[x,y,z,w]}{(xy - zw)}\right)$$

is given by

$$V(g, \partial_x g, \partial_y g, \partial_z g, \partial_w g) = V(xy - zw, y, x, -w, -z) = V(x,y,z,w) = \{(x,y,z,w)\},$$

where the last equality holds by maximality of (x,y,z,w) in $k[x,y,z,w]$. Hence, all points of X are regular except for (x,y,z,w) (corresponding to the origin in $V(xy - zw) \subseteq \mathbb{A}^4_k$).

(3)(Maxence) Consider $R = k[x,y,z,w]$. Let $I = (x,y,z,w)$, $I' = (x,z)$ and $J = (xy - zw)$. We consider the strict transform St_J (resp. St'_J) of $V(J) = X$ at I (resp. I') in $\mathbb{A}^4_k$. We denote these schemes as respectively X_1 and X_2. We know that X_1 (resp. X_2) is the closed subscheme $V_+(\bigoplus_n I^n \cap J)$ of Bl_I (resp. the closed subscheme $V_+(\bigoplus_n I'^n \cap J)$ of $\operatorname{Bl}_{I'}$).
Notice that $\operatorname{Bl}_I = \operatorname{Proj}(R[X,Y,Z,W]/\tilde{I})$ and $\operatorname{Bl}_{I'} = \operatorname{Proj}(R[X,Z]/\tilde{I}')$ where

$$\tilde{I} = (yX - xY, zX - xZ, wX - xW, yZ - zY, yW - wY, zW - wZ) \text{ and } \tilde{I}' = (zX - xZ).$$

So, the preimage of the ideal $\bigoplus_n I^n \cap J$ by the natural surjection is given by the ideal $K = \tilde{I} + (xy - zw, xY - zW, XY - ZW)$. Indeed, it must be generated in $R[X,Y,Z,W]$ by homogeneous polynomials with degree less or equal to 2 with respect to the variables X,Y,Z,W whose image is send to the generator of J which has degree 2. These generators are enough since every elements f in I^n has monomials of at least degree n and if $f \in J$, then $f = g \cdot (xy - zw)$. Since $xy - zw$ is of degree 2, the polynomial g must be of degree $n-2$, hence $g \in I^{n-2}$. So for every element in $I^n \cap J$ with $n \geq 3$ can be reach using generators of K. In the same way the preimage of $\bigoplus_n I'^n \cap J$ by the natural surjection is the ideal $K' = \tilde{I}' + (xy - zw, yX - wZ)$.
That is,

$$X_1 = \operatorname{Proj}(R[X,Y,Z,W]/K) \text{ and } X_2 = \operatorname{Proj}(R[X,Z]/K').$$

For X_1 on $D_+(X)$, we have $\mathcal{O}_{X_1}(D_+(X)) = k[x, s_1, s_2, s_3]/(s_1 - s_2 s_3)$ by simplifying the equations of K. And by the criterion, the affine open subset $D_+(X)$ of X_1 is regular. The same result holds for $D_+(Y)$,$D_+(Z)$ and $D_+(W)$ by symetry of the variables. Hence X_1 is regular.
For X_2 on $D_+(X)$, we have $\mathcal{O}_{X_2}(D_+(X)) = k[x,w,s]$ by simplying equations of K', and so $D_+(X) = \mathbb{A}^3_k$ which is regular. The same result holds for $D_+(Z)$ by symetry. Hence X_2 is regular.

(4)(Maxence) We want to compute the fiber of $f_1 : X_1 \to X$ and $f_2 : X_2 \to X$ over (x, y, z, w).
First, the residue field of $(x, y, z, w) \in X$ is simply k by exactness of localization, so for $i = 1, 2$, we need to compute the fibred product $X_i \times_X \mathrm{Spec}(k)$.
Hence, if we denote $A = k[x, y, z, w](xy - zw)$ we have

$$X_1 \times_X \mathrm{Spec}(k) = \mathrm{Proj}\left(\frac{A[X,Y,Z,W]}{K} \otimes_A k\right) \text{ and } X_2 \times_X \mathrm{Spec}(k) = \mathrm{Proj}\left(\frac{A[X,Z]'}{K} \otimes_A k\right).$$

Looking at these tensor products, by using A-linearity all relations given by K vanish except $XY - ZW = 0$ in the residue field of (x, y, z, w) by its definition. The same holds for K' but here all its relations vanish.
It yields that

$$X_1 \times_X \mathrm{Spec}(k) = \mathrm{Proj}(k[X,Y,Z,W]/(XY - ZW)) = \mathbb{P}^1_k \times_{\mathrm{Spec}(k)} \mathbb{P}^1_k$$

and

$$X_2 \times_X \mathrm{Spec}(k) = \mathrm{Proj}(k[X,Z]) = \mathbb{P}^1_k.$$

□

Exercise 4. *Introduction to derivations.* Let R be a ring and $R \to A$ an R-algebra. Let M be an A-module.
Definition. An *R-linear derivation* is a map of A-modules

$$d \colon A \to M$$

such that for all $f, g \in A$

$$d(fg) = fd(g) + gd(f) \quad \text{(Leibniz rule)}$$

We denote this set by $\mathrm{Der}_R(A, M)$.

(1) Show that $d(r) = 0$ for any $r \in R$.
(2) Consider $A \oplus M$ as a ring with multiplication

$$(a, m)(a', m') = (aa', am' + a'm)$$

Denote this ring by $A \oplus_0 M$. Show that there is a one to one correspondence between $\mathrm{Der}_R(A, M)$ and morphisms of R-algebras morphisms

$$A \to A \oplus_0 M$$

that are sections of the projection $A \oplus_0 M \to A$. Why these maps are all uninteresting if we forget about nilpotents?
(3) Show that $A[\epsilon] := A[t]/t^2$ is isomorphic to $A \oplus_0 A$.
(4) Let $X \to \mathrm{Spec}(k)$ a scheme over a field k. Let $x : \mathrm{Spec}(k) \to X$ be a k-rational point. We see $k(x)$ as an $\mathcal{O}_{X,x}$-algebra by the quotient map. Show that there are identifications

$$\mathrm{Der}_k(\mathcal{O}_{X,x}, k(x)) \cong \mathrm{Vect}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)) \cong \mathrm{Sch}_{k,x}(\mathrm{Spec}(k[\epsilon], X))$$

where $\mathrm{Sch}_{k,x}(\mathrm{Spec}(k[\epsilon], X))$ denotes k-schemes maps[2] that sends the point of $\mathrm{Spec}(k[\epsilon])$ to x

Solution Key. (1) We have $d(1) = d(1^2) = 2d(1)$ implying that $d(1) = 0$. Now using R-linearity $d(r) = rd(1) = 0$ for any $r \in R$.

(2-3) Inspection from the definition of the law. Note that $0 \oplus M$ is a square zero ideal. So all maps corresponding to derivations by the correspondence are identified when taking the reduction (are all $A_{red} \to A_{red}$)

(4) First we note that as x is a k-rational point we have a k-algebra section of the surjection $\mathcal{O}_{X,x} \to k(x)$. Using this, we may write

$$\mathcal{O}_{X,x} = k(x) \oplus \mathfrak{m}_x$$

as a direct sum of k-vector spaces. Note that a k-derivation $d\colon \mathcal{O}_{X,x} \to k(x)$ will have to send the first component to zero by the first point of the exercise. Note also that if $f, g \in \mathfrak{m}_x$ then $d(fg) = f(x)d(g) + g(x)d(f) = 0$, because we have $f(x) = g(x) = 0$. Therefore, any derivation necessarily factors through $\mathfrak{m}_x/\mathfrak{m}_x^2$. It is therefore also sufficient for a k-derivation $\mathcal{O}_{X,x} \to k(x)$ to define a k-linear map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k(x)$. This shows the first isomorphism.

For the last one, note that a k-scheme morphism from $\mathrm{Spec}(k[\epsilon])$ to X sending the point to x is equivalent to the data of a local k-algebra map $\mathcal{O}_{X,x} \to k[\epsilon]$. Projecting to $k\epsilon$ and then using the identification $k \cong k(x)$, it defines a derivation $\mathcal{O}_{X,x} \to k(x)$. The other way around, given $d\colon \mathcal{O}_{X,x} \to k(x)$, we define a k-algebra map $\mathcal{O}_{X,x} \to k[\epsilon]$ by (ev_x, d).

□

[2]meaning that the composition $\mathrm{Spec}(k[\epsilon]) \to X \to \mathrm{Spec}(k)$ is the one associated to $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ being the inclusion.