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AG II - Schemes and sheaves

Solutions – week 5

Exercise 1 and 2 are meant to prove Chevalley's theorem.

Exercise 1. *Projection from affine spaces.* Let R be a ring.

(1) Show that
$$\pi\colon \operatorname{Spec}(R[t]) \to \operatorname{Spec}(R)$$
is open. More precisely, if $f = \sum a_i t^i$ show that
$$\pi\left(D(f(t))\right) = \bigcup_i D(a_i).$$

(2) Let $g(t) \in R[t]$ be a monic polynomial and $f(t) \in R[t]$. Remark that $R[t]/g(t)$ is a free R-module of rank $\deg(g)$. Let $\chi(X) = \sum_i^n r_i X^i$ be the characteristic polynomial of the multiplication by $f(t)$ on $R[t]/g(t)$. Show that
$$\pi(D(f) \cap V(g)) = \bigcup_i^{n-1} D(r_i).$$

Solution key. (1) Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $\mathfrak{p} \in \pi(D(f(t)))$ if and only if $k(\mathfrak{p})[t]_{f(t)} \neq 0$ if and only if $f(t) \neq 0$ in $k(\mathfrak{p})[t]$ if and only if there is some i such that $a_i \notin \mathfrak{p}$.

(2) Let $\mathfrak{p} = \mathfrak{q} \cap R$ with $\mathfrak{q} \in D(f) \cap V(g)$ in the image. So we have a map $k(\mathfrak{p})[t]/(g(t)) \to k(\mathfrak{q})$. Note the following fact: by Cayley-Hamilton f is nilpotent in $k(\mathfrak{p})[t]/(g(t))$ if and only if $\mathfrak{p} \in V(r_0, \ldots, r_{n-1})$.

Note also that $f \neq 0$ in $k(\mathfrak{q})$ because $\mathfrak{q} \in D(f)$. So f is not nilpotent in $k(\mathfrak{p})[t]/(g(t))$ and therefore $\mathfrak{p} \in \bigcup_{i=1}^{n1} D(r_i)$.

Reciprocally if $\mathfrak{p} \in \bigcup_{i=1}^{n1} D(r_i)$, the by the above argument f is not nilpotent in $k(\mathfrak{p})[t]/(g(t))$. Therefore there is some $\mathfrak{q} \notin f$ in $k(\mathfrak{p})[t]/(g(t))$ meaning that $\mathfrak{q} \in D(f) \cap V(g)$, which is therefore sent to $\mathfrak{p}$.

□

Exercise 2. *Chevalley's theorem.* Let X be a Noetherian topological space. A subset $T \subset X$ is called *constructible* if it can be written as a finite union of sets of the form $U \cap V^c$ where U and V are open sets.

(1) Show that if $X = \operatorname{Spec}(R)$ for a Noetherian ring R, a subset is constructible if and only if it can be written as a finite union of subsets of the form $D(f) \cap V(g_1, \ldots, g_m)$ with $f, g_1, \ldots, g_m \in R$.

(2) Show using the above exercise that
$$\pi\colon \operatorname{Spec}(R[t]) \to \operatorname{Spec}(R)$$
sends constructible subsets to construtible subsets.

Hint: Show by induction on $\sum_i \deg(g_i)$ that if $f, g_1, \dots, g_m \in R[t]$ are polynomials, the image of $D(f) \cap V(g_1, \dots, g_m)$ is constructible. To conduct the induction step, consider α the leading coefficient of g_1. Break down the study on the open and closed $D(\alpha)$ and $V(\alpha)$ to reduce the sum of the degrees.

(3) Deduce *Chevalley's theorem.* Let $f\colon X \to Y$ be a finite type morphism between Noetherian schemes. Then f sends constructible subsets to constructible subsets.[1]

Solution key. (2) Note that we already know two cases. Namely the case $D(f)$ and the case $D(f) \cap V(g)$ where g is monic. We proceed by induction on the sum of the degrees of g_i – also we order them such that they have increasing degrees. Let c be the dominant coefficient of g_1. We have

$$\operatorname{Spec}(R[t]) = \operatorname{Spec}(R/c[t]) \sqcup \operatorname{Spec}(R_c[t]).$$

In the first the image of g is of degree striclty less. So induction goes.

Also, note that g_1 is monic in $\operatorname{Spec}(R_c[t])$. If $n = 1$, we are in an already dealt situation. If not let

$$g_2' = g_2 - t^{d_2 - d_1}(c'/c)g_1$$

where c' is the leading coefficient of g_1. Then

$$D(f) \cap V(g_1, g_2, \dots, g_n) = D(f) \cap V(g_1, g_2', \dots, g_n).$$

But now the sum of degrees of the list lowers giving the claim by induction.

(3) We can reduce to the affine case where we can reduce to

$$R \to R[t_1, \dots, t_n] \to S$$

where the last map is surjective. The first arow induces on Spec a map which preserves constructibility by the above and the second also because it is a closed immersion

□

Remark. In general the topological image of a morphism of schemes can fail to be open or closed but in cases where Chevalley's theorem applies, it tells that it still not too far from it and manageable. In particular one can endow the image with a scheme structure.

Exercise 3. *An application of Chevalley's theorem.* Let $f\colon X \to Y$ be a finite type dominant map between Noetherian schemes with Y irreducible. Use Chevalley's theorem to show that the topological image $f(X)$ contains an open set.

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[1]The generalization to non-Noetherian settings requires more careful definitions, but once these definitions are addressed the proof is the same.

Solution key. The image being dense contains the generic point of Y, therefore, $\eta_Y \in U \cap Z \subset f(X)$ because the topological image $f(X)$ is constructible, for some open U and closed Z of Y. But if $\eta_Y \in Z$ then we see that $Z = Y$ □

Exercise 4. *Nullstellensatz* via *Chevalley.* Let k be a field and $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_n]$.

(1) Show by contradiction that $\mathfrak{p}_i = k[x_i] \cap \mathfrak{m}$ is maximal (so $\neq 0$) for each $i = 1, \ldots, n$.
Hint: If $\mathfrak{p}_i = 0$, *we have a dominant map* $\mathrm{Spec}(k(\mathfrak{m})) \to \mathbb{A}^1_k$.

The above is called *Zariski's lemma* and is the key to Nullstellensatz. Deduce from the lemma proved in item (1) the following direct consequences.

(2) Deduce the *Nullstellensatz*, meaning that $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k, and that
$$\mathfrak{m} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_n).$$
(3) Let $A \to B$ a k-algebra map between finite type k-algebras. Show that $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ carries closed points to closed points.
(4) Deduce that any finite type finite type k-algebra A is *Jacobson*, meaning that the nilradical (intersection of all primes, see week 3 exercise 1) of A is equal to the intersection of maximal ideals of A.
Hint: for f not nilpotent, use the preceding point with $A \to A_f$.

Solution key.

(1) Let $\mathfrak{m}$ be maximal in $k[x_1, \ldots, x_n]$. Suppose by contradiction that $\mathfrak{p}_i = k[x_i] \cap \mathfrak{m}^2$ is not maximal, because it is prime, we have $\mathfrak{p}_i = (0)$. The image of the map $\mathrm{Spec}(k(\mathfrak{m})) \to \mathbb{A}^1_{k,x_i}$ is $\mathfrak{p}_i$. But by Chevalley, the image of the map $\mathrm{Spec}(k(\mathfrak{m})) \to \mathbb{A}^1_{k,x_i}$ is constructible, but by our hypothesis, also contains the generic point, and therefore contains an open set. But an open in $\mathbb{A}^1_k$ contains infinitely many points, way much than our singleton $\{\mathfrak{p}_i\}$, leading to a contradiction.
(2) We see by successive quotients, because each $\mathfrak{p}_i$ is maximal in $k[x_i]$, that $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ is maximal. But has it is contained in $\mathfrak{m}$ we have our claimed equality. Also if we denote by $k[x_i]/(\mathfrak{p}_i) = k(\alpha_i)$ where α_i is therefore an algebraic element over k. Then
$$k[x_1, \ldots, x_n]/\mathfrak{m} = k(\alpha_1, \ldots, \alpha_n)$$
and therefore a finite extension.
(3) First, note that from the last point, we deduce that every residue field of a finite type k-algebra at a closed point is a finite extension of k. Let $\mathfrak{m}$ be maximal in $\mathrm{Spec}(B)$. Then we have injections
$$k \to A/f(\mathfrak{m}) \to B/\mathfrak{m}.$$
Because $B/\mathfrak{m}$ is finite dimensional over k, so is $A/f(\mathfrak{m})$. But then the multiplication by every non-zero element is injective, but then

[2] This is the projection to the i-th coordinate.

surjective because it is a self of a finite dimensional k-vector space. We conclude that $A/f(\mathfrak{m})$ is a field, leading to the desired conclusion.

(4) It suffices to show that every element which is not nilpotent is contained in some maximal ideal. If f is not nilpotent, then $A_f \neq 0$. So there is a maximal ideal in A_f. By the previous point, the preimage of this ideal is maximal in A, concluding.

□

Exercise 5. *Integrality/reducedness of* Proj. Let B be an $\mathbb{N}$-graded integral/reduced ring. Show that $\mathrm{Proj}(B)$ is an integral/reduced scheme.

Proof. We need to show that $\mathrm{Proj}(B)$ is reduced (and irreducible). Note that for any homogeneous element $b \in B$, the ring $B_{(b)}$ is reduced if B is reduced. Also, note that if B is a domain (0) is prime and is the unique generic point of the space $\mathrm{Proj}(B)$. □

Exercise 6. *Fibers.*

(1) Compute the fibers of the morphism

$$\mathrm{Spec}(\mathbb{Z}[x, y, z]/(2zx + 9y^2)) \to \mathrm{Spec}(\mathbb{Z}).$$

Which fiber is reduced ? Which fiber is integral ?

(2) Compute the fibers of the morphism, where p is a prime number

$$\mathrm{Spec}(\mathbb{Z}[x, y]/(xy^2 + p)) \to \mathrm{Spec}(\mathbb{Z}).$$

Which fiber is reduced ? Which fiber is integral ?

Solution key. (1) The fiber over 2 is not reduced. The fiber over 3 is reduced but not integral. It is integral over any other prime by Eisenstein criterion.

(2) The fiber over p is not reduced and not irreducible. Othewise it is isomorphic to $k[x, y, y^{-1}]$ where k is a prime field not equal to $\mathbb{F}_p$.

□

Exercise 7. *Fibers (2).*

(1) Show that for the morphism $\mathrm{Spec}(k[x, y]/(xy)) \to \mathrm{Spec}(k[x])$, induced by the obvious map $k[x] \to k[x, y]/(xy)$, every fiber is irreducible, although the $\mathrm{Spec}(k[x, y]/(xy))$ is not.

(2) Show that for the morphism $\mathrm{Spec}(\mathbb{Q}[t]) \to \mathrm{Spec}(\mathbb{Q}[t])$ induced by $t \mapsto t^2$ there are infinitely many closed points with irreducible fibers and infinitely many closed points with non-irreducible fibers.