

Solutions – week 4

Some notation. We introduce some notation needed for exercise 1 below. Let A and B be \mathbb{N} -graded rings. Let $d \geq 1$. We define the graded ring

$$A^{(d)} = \bigoplus_{n \geq 0} A_{nd}$$

and $v_d: A^{(d)} \rightarrow A$ for the canonical inclusion as a subring. This subring is called the d -Veronese subring.

We say that a ring map $\psi: A \rightarrow B$ is *homogeneous of degree d* if for $n \geq 0$ A_n maps to B_{dn} ¹. For example for the usual grading on $\mathbb{Z}[x]$ show that $x \mapsto x^d$ is homogeneous of degree d . Also, v_d is homogeneous of degree d .

Exercise 1. *Functoriality of Proj.* Let A and B be \mathbb{N} -graded rings. Let $\psi: A \rightarrow B$ be an homogeneous map of degree d for some $d \geq 1$.

To the contrary of Spec, the functoriality of Proj is not evident. The reason is that $\psi^{-1}(\mathfrak{p})$ for a prime $\mathfrak{p} \in \text{Proj}(B)$ may contain the irrelevant ideal A_+ .

- (1) Show that

$$U(\psi) = \{\mathfrak{p} \in \text{Proj}(B) \mid \psi(A_+) \not\subset \mathfrak{p}\}$$

is open. Namely show that it is the union of opens $D_+(\psi(f))$ for all homogeneous $f \in A_+$.

- (2) Find an example where $U(\psi)$ is a non-empty open strict subspace of $\text{Proj}(B)$.
- (3) Show that ψ^{-1} defines a map of schemes $r_\psi: U(\psi) \rightarrow \text{Proj}(A)$. Do this by defining a map $D_+(\psi(f)) \rightarrow D_+(f)$ for all homogeneous $f \in A_+$ and then glue.
- (4) Show that if there exists a k_0 such that for all $k \geq k_0$ the map $A_k \rightarrow B_{dk}$ is surjective then $U(\psi) = \text{Proj}(B)$. Show moreover that in this case r_ψ is a topological closed embedding with image $V_+(\ker(\psi))$.
- (5) Show that if there exists a k_0 such that for all $k \geq k_0$ the map $A_k \rightarrow B_{dk}$ is an isomorphism then r_ψ is an isomorphism.
- (6) Deduce that for any $d \geq 1$ and for $v_d: A^{(d)} \rightarrow A$ the map r_{v_d} is an isomorphism.

Solution key. (2) For example $\iota: \mathbb{Z}[x, y] \rightarrow \mathbb{Z}[x, y, z]$ the inclusion. Then $U(\iota) = D_+(x) \cup D_+(y)$. In particular $(X, Y) = [0 : 0 : 1]$ is not in this open.

- (3) We use that $U(\psi)$ is a gluing

$$U(\psi) = \bigcup_{a \in A_+ \text{ hom.}} \text{Spec}(B_{(\psi(a))})$$

¹Note that this means that the map factors through the d -Veronese subring.

with gluing data is given by, at the dual level of ring of functions by the natural maps

$$\begin{array}{ccc} B_{(\psi(a))} & & \\ & \searrow & \\ B_{(\psi(a'))} & \longrightarrow & B_{(\psi(aa'))} \end{array}$$

So to define a map out of $U(\psi)$, it suffices to define a map from each $\text{Spec}(B_{(\psi(a))})$. To do this we glue the map induced by ψ

$$\text{Spec}(B_{(\psi(a))}) \rightarrow \text{Spec}(A_{(a)}).$$

This glues because the following commutes as every map is giving by the natural extension-restriction of ψ to these rings

$$\begin{array}{ccccc} A_{(a)} & \longrightarrow & B_{(\psi(a))} & & \\ & \searrow & & \searrow & \\ & & A_{(aa')} & \longrightarrow & B_{(\psi(aa'))} \\ & \nearrow & & \nearrow & \\ A_{(a')} & \longrightarrow & B_{(\psi(a'))} & & \end{array}$$

- (4) We first show that $U(\psi) = \text{Proj}(B)$. Let $\mathfrak{p} \in D_+(b)$ any point of $\text{Proj}(B)$ and some homogeneous $b \in B_+$. Note that b^{dk_0} is in the image of ψ by hypothesis, which concludes.

To show that the map is a closed immersion, it suffices to show that locally

$$\text{Spec}(B_{(\psi(a))}) \rightarrow \text{Spec}(A_{(a)})$$

is a closed immersion. But therefore it suffices to show that the underlying map of rings is a surjection. Because $D_+(a) = D_+(a^N)$ for any $N \geq 1$ we can suppose that $\deg(a) \geq k_0$. But then an element of $B_{(\psi(a))}$ is of the form $\frac{b}{\psi(a)^n}$ with $\deg(b) = dn \deg(a)$. But as $A_{n \deg(a)} \rightarrow B_{dn \deg(a)}$ is surjective by assumption, we win.

Now we are left to show that the image is $V_+(\ker(\psi))$. It suffices to show that it's the image when intersecting to every $D_+(a)$. But $D_+(a) \cap V_+(\ker(\psi)) = V(\ker(\psi_{(a)}))$, which concludes.

- (6) Same local trick. It suffices to show that it's locally an isomorphism. Enlarging degrees is again harmless.

□

Exercise 2. *Basic properties of \mathbb{P}_A^n .*

Let A be a ring and $A[x_0, \dots, x_n]$ the polynomial ring in $n+1$ variables over A . We define $\mathbb{P}_A^n := \text{Proj}(A[x_0, \dots, x_n])$.

- (1) Show that $\Gamma(\mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^n}) = A$.

Note: this is a first instance of a more general fact about projective varieties and can be thought of as an algebraic instance of the maximum modulus principle in complex analysis. See (3).

- (2) Assume that $A = k$, where k is an algebraically closed field. Show that the closed points of \mathbb{P}_k^n are identified with $(n + 1)$ -tuples $[a_0 : \dots : a_n]$ satisfying the following properties:
- $a_i \in k$ for all i ,
 - not all a_i are 0, and
 - two $(n + 1)$ -tuples $[a_0 : \dots : a_n]$ and $[b_0 : \dots : b_n]$ are identified if there exists $c \in k^*$ such that $b_i = c \cdot a_i$ for all i .
- In other words, the points are identified with $(k^{n+1} \setminus 0)/k^\times$, *i.e.* linear subspaces of dimension 1 of k^{n+1} .
- (3) Let $A = k$ be a field and B be a k -algebra. Show that every morphism of k -schemes $\mathbb{P}_k^n \rightarrow \text{Spec}(B)$ is constant at the level of topological spaces with image a closed point which is k -rational.
- (4) Let d be a positive integer and set $m := \binom{n+d}{d} - 1$. Use Exercise 1 and monomials of degree d to define an everywhere defined morphism², that we call a d -th Veronese embedding of \mathbb{P}_A^n

$$\psi_d: \mathbb{P}_A^n \rightarrow \mathbb{P}_A^m.$$

- (5) Let k be an algebraically closed field. Describe the image of a second Veronese embedding $\psi_2: \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^2$. Furthermore, using part (2), describe the closed points of the image as triples $[a_0 : a_1 : a_2]$.

This was an hand-in exercise a previous year and solutions are attributed to students who wrote them.

Solution key. (1) (*Kangyeon*) Let $s \in \Gamma(\mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^n})$. Then the restriction $s|_{D_+(x_i)}$ is induced from $f \in \mathcal{O}_{\text{Spec}(S_{(x_i)})}(\text{Spec}(S_{(x_i)}) \cong S_{(x_i)} \cong A[y_1, \dots, y_n]$ in the following way: if $f(y_1, \dots, y_n) \in A[y_1, \dots, y_n]$, then any $\mathfrak{p} \in D_+(x_i)$ is sent to $f(x_0/x_i, \dots, x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_n/x_i) \in S_{(\mathfrak{p})}$. Thus it is of the form $g_i(x_0, x_1, \dots, x_n)/x_i^{m_i}$ where $g_i \in A[x_0, x_1, \dots, x_n]$ is homogeneous with $\deg g_i = m_i \in \mathbb{Z}_{\geq 0}$. Now for $\mathfrak{p} \in D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j)$, we must have

$$\frac{g_i(x_0, x_1, \dots, x_n)}{x_i^{m_i}} = \frac{g_j(x_0, x_1, \dots, x_n)}{x_j^{m_j}}$$

in $S_{(\mathfrak{p})}$. As $S_{(x_i x_j)} \cong \mathcal{O}_{\text{Spec}(S_{(x_i x_j)})}(\text{Spec}(S_{(x_i x_j)}))$ is isomorphic to $\mathcal{O}_{\text{Proj}(S)}(D_+(x_i x_j))$, we see that above fractions are equal in $S_{(x_i x_j)}$. In other words,

$$(x_i x_j)^n (g_i(x_0, x_1, \dots, x_n) x_j^{m_j} - g_j(x_0, x_1, \dots, x_n) x_i^{m_i}) = 0$$

in S . This forces

$$g_i(x_0, x_1, \dots, x_n) x_j^{m_j} = g_j(x_0, x_1, \dots, x_n) x_i^{m_i}$$

as $x_i x_j$ is not a zero-divisor, and we see that $x_i^{m_i} \mid g_i(x_0, x_1, \dots, x_n)$. Thus by degree argument, $g_i(x_0, x_1, \dots, x_n)/x_i^{m_i} \in A$ and they are all equal by (1), so that s is induced from $a \in A$ in a way that $s(\mathfrak{p}) = a/1 \in S_{(\mathfrak{p})}$. This map is obviously a ring homomorphism,

²Induce a map using Proj from an homogeneous map of degree d that sends monomials of degree 1 to all monomials of degree d .

which we have just shown to be surjective. Moreover different $a \in A$ give different element of $\Gamma(\mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^n})$, for if $a/1 = 0 \in S_{(\mathfrak{p})}$ for every $\mathfrak{p} \in \text{Proj}(S)$, then the annihilator I of a is not contained in any $\mathfrak{p} \in \text{Proj}(S)$, so that $I \in V(S_+)$, and in particular $x_0 a = 0$ in S , which forces $a = 0$. This we have an isomorphism $A \rightarrow \Gamma(\mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^n})$.

- (2) (*Mathis*) Let $x \in \mathbb{P}_A^n$ be a closed point. x is contained in some $D^+(x_i)$ which is affine, so that x corresponds to a maximal ideal of $k[x_0, \dots, \widehat{x}_i, \dots, x_n] \xrightarrow{\varphi_i} k[x_0, \dots, x_n, x_i^{-1}]_0$. By weak nullstellensatz, this corresponds to an ideal $(x_0 - a_0, \dots, x_n - a_n)$ for some $a_j \in k$, $j \in \{0, \dots, n\} \setminus \{i\}$. This corresponds via φ_i to the homogeneous ideal $(x_0/x_i - a_0, \dots, x_n/x_i - a_n)$. We thus associate the tuple $[a_0, \dots, 1, \dots, a_n]$ to x with the 1 in the i -th coordinate. Suppose we associate to it another tuple $[b_0, \dots, 1, \dots, b_n]$ with 1 in the j -th coordinate for $x \in D^+(x_j)$. We have

$$\begin{aligned} \varphi_j^{-1} \circ \iota_{ji}^{-1} \circ \iota_{ij} \circ \varphi_i(x_0 - a_0, \dots, x_n - a_n) &= (x_0 - b_0, \dots, x_n - b_n) \\ &= \varphi_j^{-1} \circ \iota_{ji}^{-1}(x_0/x_i - a_0, \dots, x_n/x_i - a_n) \\ &= \varphi_j^{-1} \circ \iota_{ji}^{-1}((x_0/x_j)a_j - a_0, \dots, (x_n/x_j)a_j - a_n, (x_i/x_j)a_j - 1) \\ &= (a_j x_0 - a_0, \dots, a_j x_n - a_n, a_j x_i - 1) \\ &= (x_0 - a_0/a_j, \dots, x_n - a_n/a_j) \end{aligned}$$

We deduce that $b_k = a_k/a_j$ (note we could divide by a_j since $x \in D^+(x_j)$ is equivalent to $a_j \neq 0$). This shows that the assignment of x to the class of $[a_0, \dots, 1, \dots, a_n]$ under multiplication by k^\times is well defined (ie independent of the chosen affine containing x). By construction not all coordinates are 0 (the non-zero ones correspond to the affines in which x belongs) and they all belong to k . The assignment is injective since if x, y are assigned to the same tuple $[a_0, \dots, a_n]$, then locally in some common affine $D(x_i)$ (such an affine must exist by hypothesis since the set of non-zero a_i corresponds to the affines containing x, y), x and y both correspond to the same maximal ideal so are equal. It is surjective for given a tuple $[a_0, \dots, a_n]$, wlog we have $a_0 \neq 0$ so we can take a representative $[1, b_1, \dots, b_n]$, and taking the closed point of $D_+(x_0)$ corresponding to $(x_1 - b_1, \dots, x_n - b_n)$ gives a point associated to this tuple class

- (3) (*Dévo*) Using the Spec and global sections adjunction, we know that maps $\mathbb{P}_k^n \rightarrow \text{Spec}(B)$ are in natural bijection with maps $B \rightarrow \mathcal{O}_{\mathbb{P}_k^n}(\mathbb{P}_k^n)$, and by point **(2)**, the global sections are isomorphic to k . Putting this together with the fact that we are working with schemes over $\text{Spec}(k)$, we see that the set of maps $\mathbb{P}_k^n \rightarrow \text{Spec}(B)$ is in natural bijection with section of the k -algebra structure map $k \rightarrow B$. Now we know that these are exactly the k -rational points B . Finally, using the naturality of the isomorphisms exhibiting Spec as adjoint to global sections, we get that the maps $\mathbb{P}_k^n \rightarrow \text{Spec}(B)$ factor through $\text{Spec}(k)$ and must map to k -rational points of B .
- (4) (*Kangyeon*) There are $m+1$ -number of monomials of degree d in $S = A[x_0, \dots, x_n]$. Thus we have a homogeneous ring homomorphism

of degree d $\varphi : S' = A[x_0, \dots, x_m] \rightarrow S$ by sending each x_i to a monomial of degree d . Then for $k \geq 0$ each $S'_k \rightarrow S_{kd}$ is surjective; each monomial of degree kd can be split into k number of monomials of degree d , which are images of x_0, \dots, x_m , so the original monomial is in the image of S'_k . If $k = 0$ this is trivial. Thus from this we induce a morphism of schemes from \mathbb{P}_A^n to \mathbb{P}_A^m .

- (5) (*Mathis*) We use homogeneous coordinates according to part 3 of this exercise. We pick the choice of bijection F as in the previous part to be $x_0 \mapsto x_0^2, x_1 \mapsto x_0x_1, x_2 \mapsto x_1^2$. Let $x \in \mathbb{P}_k^1$ be described by homogeneous coordinates $[a : b]$. It thus corresponds to the homogeneous ideal $(ax_1 - bx_0)$ in $A[x_0, x_1]$. The degree 2 part of this ideal is $(ax_1^2 - bx_0x_1, ax_1x_0 - bx_0^2)$ so that it maps through ψ_2 to $(ax_2 - bx_1, ax_1 - bx_0)$. If $a \neq 0$ this is $(ax_2 - b^2/ax_0, ax_1 - bx_0)$ which corresponds to the coordinates $[a : b : b^2/a] = [a^2 : ab : b^2]$. If $b \neq 0$ we have $(ax_2 - bx_1, ax_0 - a^2/bx_1)$ which corresponds to $[a^2/b : a : b] = [a^2 : ab : b^2]$. Similarly $c \neq 0$ gives the coordinates $[a^2 : ab : b^2]$.

□

Exercise 3. *Finite covers of $\mathbb{P}_{\mathbb{C}}^1$.* Let $n \geq 1$. Consider the self map c_n of \mathbb{C} -schemes on $\text{Proj}(\mathbb{C}[x, y]) = \mathbb{P}_{\mathbb{C}}^1$ induced by Proj from the \mathbb{C} -algebra map $x \mapsto x^n$ and $y \mapsto y^n$ on $\mathbb{C}[x, y]$.

- (1) Compute the preimage by this map of $D_+(x)$ and $D_+(y)$, show it's affine.
- (2) Compute all the fibers of the map.

Solution key. (1) The preimage of $D_+(x)$ and $D_+(y)$ is $D_+(x^n) = D_+(x)$ and $D_+(y^n) = D_+(y)$. The map on ring of functions is given the \mathbb{C} -algebra map given by sending $\frac{y}{x} \mapsto \left(\frac{y}{x}\right)^n$ and $\frac{x}{y} \mapsto \left(\frac{x}{y}\right)^n$. Now note that $\mathbb{C}[t^n] \subset \mathbb{C}[t]$ is finite free of degree n . A basis being $1, t, \dots, t^{n-1}$.

(2) The generic fiber is $\mathbb{C}\left(\frac{x}{y}\right)$ because the generic point's unique preimage is the generic point for dimension reasons. But let us also deduce as follows. Locally this amounts to compute the tensor product=pushout of

$$\begin{array}{c} \mathbb{C}[x/y] \xrightarrow{x/y \mapsto (x/y)^n} \mathbb{C}[x/y] \\ \downarrow \\ \mathbb{C}(x/y) \end{array}$$

But note that localizing $\mathbb{C}[x/y]$ at the multiplicative $(\mathbb{C}[x/y])^n \setminus 0$ (where the power n here means elements of this ring that are the n -th power) is the same as localizing by the multiplicative subset $\mathbb{C}[x, y] \setminus 0$. Indeed inverting an element or it's n -th power is the same. As for the fiber of closed points, say $(t - \lambda)$ where $t = x/y$ or y/x seen in $D_+(x)$ or $D_+(y)$, we get

$$\text{Spec}(\mathbb{C}[t]/(t^n - \lambda)).$$

So if $\lambda = 0$, we get a single non-reduced point and n -copies of $\text{Spec}(\mathbb{C})$ otherwise. \square

Exercise 4. *Ramifications of a self map of \mathbb{P}^1 .*

- We say that a map of schemes $f: X \rightarrow Y$ is *finite locally free* if there is a covering of Y by open affines $\text{Spec}(A_i)$, with affine preimage $\text{Spec}(B_i)$, such that induced map $A_i \rightarrow B_i$ turns B_i into a finite free A_i -module. When for every i the dimension of B_i is the same, say d , we say that the map is *finite locally free of degree d* .
 - We say that a finite locally free map $X \rightarrow Y$ is *ramified at $y \in Y$* if the geometric fiber $X_{\bar{y}}$ is not reduced.
- (1) Show that the self map c_n from exercise 3 is finite locally free of degree n and identify its ramification points.
 - (2) Let R be a ring. Show that the map induced on $\mathbb{P}_R^1 = \text{Proj}(R[x, y])$ by the R -algebra self map $x \mapsto ax + by$ and $y \mapsto cx + dy$ is an automorphism if $ad - bc \in R^\times$. We denote this map $m_{(a,b,c,d)}$.
If $R = \mathbb{C}$ and if we identify $\mathbb{P}_{\mathbb{C}}^1(\mathbb{C}) = \mathbb{C} \cup \infty$, how is this map expressed on \mathbb{C} -rational points?
 - (3) Consider the composition

$$\mathbb{P}_{\mathbb{C}}^1 \xrightarrow{c_n} \mathbb{P}_{\mathbb{C}}^1 \xrightarrow{m_{(1,-1,1,1)}} \mathbb{P}_{\mathbb{C}}^1 \xrightarrow{c_2} \mathbb{P}_{\mathbb{C}}^1.$$

Show it's finite locally free of fixed degree. What is the degree? What are the ramification points? Compute scheme theoretic fibers at all ramification points.

This was an hand-in exercise a previous year and solutions are attributed to students who wrote them.

Solution key. (1) (*Kangyeon*) The preimage of $D_+(x) \cong \text{Spec}(\mathbb{C}[x, y]_{(x)}) \cong \text{Spec}(\mathbb{C}[t])$ (by identifying $t = y/x$) under c_n is $D_+(x^n) = D_+(x) = \text{Spec}(\mathbb{C}[s])$ and the same for y . The induced morphism of \mathbb{C} -algebras $\mathbb{C}[t] \rightarrow \mathbb{C}[s]$ is given by $t \mapsto s^n$. Thus $\mathbb{C}[s]$ is freely generated by $1, s, \dots, s^{n-1}$ as $\mathbb{C}[t]$ -module, and the same for y . Thus c_n is locally finite free of degree n .

Let $\mathfrak{p} \in \mathbb{P}_{\mathbb{C}}^1$ be a point. If $\mathfrak{p} \in D_+(x) \cong \text{Spec}(\mathbb{C}[t])$, the fiber is locally of the form $\text{Spec}(\mathbb{C}[s] \otimes_{\mathbb{C}[t]} k(\mathfrak{p}))$. Since \mathfrak{p} is a closed point the form $(t - \lambda)$ or the generic point (0) , we compute each geometric fiber. In the first case, $k(\mathfrak{p}) = \mathbb{C}[t]/(t - \lambda) \cong \mathbb{C}$ is already algebraically closed, hence the tensor product is $\mathbb{C}[s]/(s^n - \lambda)$. If $\lambda = 0$ this is not reduced, and otherwise it is reduced, as $s^n - \lambda = \prod_{k=0}^{n-1} (s - \sqrt[n]{\lambda} e^{2\pi i k/n})$ is radical. For the generic point, $k(\mathfrak{p}) = \text{Frac}(\mathbb{C}[t]) = \mathbb{C}(t)$, hence the tensor product is $\mathbb{C}[s] \otimes_{\mathbb{C}[t]} \mathbb{C}(t) = (\mathbb{C}[t][u]/(u^n - t) \otimes_{\mathbb{C}[t]} \mathbb{C}(t)) = \mathbb{C}(t)[u]/(u^n - t) = \mathbb{C}(t)$, which is reduced. Similarly we can deal with $D_+(y)$, and since reducedness can be checked at the level of stalks, we see that the only ramified points correspond to $\lambda = 0$ in both cases. In other identifications,

they correspond to the prime ideals (x) and (y) , or the points $[1 : 0] = 0$ and $[0 : 1] = \infty$.

- (2) (*Kangyeon*) The R -algebra map is homogeneous of degree 1, and admits inverse if $u := ad - bc \in R^\times$. Indeed, $x \mapsto u^{-1}(dx - by)$, $y \mapsto u^{-1}(-cx + ay)$ is the inverse (which is also homogeneous of degree 1), as one can easily check by computation. Thus this surjective map induces a morphism $\mathbb{P}_R^1 \rightarrow \mathbb{P}_R^1$ by the functoriality of Proj, which has inverse induced by the inverse described above. Hence this is an automorphism.
- (3) (*Mathis*) We first show the following technical lemmas.

$f : X \rightarrow Y$ is finite locally free of rank d iff all induced maps of stalks $\mathcal{O}_{Y,f(\mathfrak{p})} \rightarrow \mathcal{O}_{X,\mathfrak{p}}$ make $\mathcal{O}_{X,\mathfrak{p}}$ into a free $\mathcal{O}_{Y,f(\mathfrak{p})}$ module of rank d

Proof. The \implies direction is trivial since localisations of free modules are free. For the converse, first note that f is finite locally free of rank d iff each $y \in Y$ has an affine open neighborhood U such that $f^{-1}(U) \rightarrow U$ is finite locally free of rank d . Thus wlog we may assume $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. Consider $f(\mathfrak{p}) \in Y$. We have that $B_{f(\mathfrak{p})} \rightarrow A_{\mathfrak{p}}$ makes $A_{\mathfrak{p}}$ into a free rank d module over $B_{f(\mathfrak{p})}$. Consider a basis $\{a_i/b_i\}_{i=1}^d \subset A_{\mathfrak{p}}$ over $B_{f(\mathfrak{p})}$. Then if we let $g = \prod_{i=1}^d b_i$, we get that A_g is free of rank d over $S^{-1}B$ with $S = f^\#(\{1, g, g^2, \dots\})$. Thus if we consider $U \subset Y$ corresponding to $S^{-1}B$ about \mathfrak{p} , preimage contains $D(g)$, and thus up to shrinking we may assume it is contained in $D(g)$. We conclude that $f : X \rightarrow Y$ is finite locally free of rank d . \square

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be finite locally free maps of rank m, n respectively. Then $g \circ f$ is finite locally free of rank mn .

Proof. Follows directly from the previous lemma: a free module C of rank m on B , which is itself a free module of rank n on A , will be a free module of rank mn on A (one can also just check multiplicativity at the level of residue field extensions). \square

Note that $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{C})$. We can now use that c_n is finite locally free of rank n , c_2 finite locally free of rank 2, and that $m_{(1,-1,1,1)}$ is finite locally free of rank 1 (since it is induced by automorphisms of rings), to deduce that $c_2 \circ m_{(1,-1,1,1)} \circ c_n$ is finite locally free of rank $2n$.

Now to compute fibers of $f = c_2 \circ m_{(1,-1,1,1)} \circ c_n$ we can note that (writing $m = m_{(1,-1,1,1)}$ for short)

$$\text{Spec}(\mathbb{C}) \times_{f, \mathbb{P}_{\mathbb{C}}^1} \mathbb{P}_{\mathbb{C}}^1 \cong ((\mathbb{P}_{\mathbb{C}}^1 \times_{c_2, \mathbb{P}_{\mathbb{C}}^1} \text{Spec}(\mathbb{C})) \times_{m, \mathbb{P}_{\mathbb{C}}^1} \mathbb{P}_{\mathbb{C}}^1) \times_{c_n, \mathbb{P}_{\mathbb{C}}^1} \mathbb{P}_{\mathbb{C}}^1$$

Thus since m is an automorphism, we can note that the points that ramify will be $c_2 \circ m([0, 1])$, $c_2 \circ m([1, 0])$ (since those ramify under c_n) as well $[0, 1]$, $[1, 0]$ (since those ramify under c_2). In $\mathbb{C} \cup \infty$ this

corresponds respectively to the points

$$\left(\frac{0-1}{0+1}\right)^2 = (-1)^2 = 1, \quad \left(\frac{\infty-1}{\infty+1}\right)^2 = (1)^2 = 1, \quad 0, \quad \infty$$

where we have slightly abused of notation. We should expect the generic point to be unramified. Indeed we can explicitly compute its fiber using the same computation as in exercise 1: pulling back through c_2 gives $\text{Spec}(\overline{\mathbb{C}(z)}) \cup \text{Spec}(\overline{\mathbb{C}(z)})$. Pulling back through m does not change the fiber structure. Finally pulling back through c_n gives $\bigsqcup_{i=1}^{2n} \text{Spec}(\overline{\mathbb{C}(z)})$.

There are thus only three ramified points: $0, 1, \infty$. It remains to compute their fibers. The fibers of 0 and ∞ under c_2 are $\text{Spec}(\mathbb{C}[x]/(x^2))$ up to isomorphism. We can further pull back through m and still preserve this fiber structure up to isomorphism. Now since $(c_2 \circ m)^{-1}(0)$ and $(c_2 \circ m)^{-1}(\infty)$ do not contain ramified points of c_n , we can pull back through c_n to get some $\lambda \neq 0$ such that $(\mathbb{P}_{\mathbb{C}}^1)_{0,f}$ is given by Spec of

$$\mathbb{C}[x]/((x-\lambda)^2) \otimes_{\mathbb{C}[x]} \mathbb{C}[x^{1/n}] \cong \mathbb{C}[x, y]/((x-\lambda)^2, y^n - x) \cong \mathbb{C}[y]/((y^n - \lambda)^2)$$

(note how this still only has length 2). A similar thing holds for $(\mathbb{P}_{\mathbb{C}}^1)_{\infty,f}$

For 1 , this is a non ramified point of c_2 and thus its fiber is $\text{Spec}(\mathbb{C}[x]/(x^2 - 1)) \cong \text{Spec}(\mathbb{C}) \cup \text{Spec}(\mathbb{C})$, corresponding to -1 and 1 . Pulling further back through m preserves the fiber structure, with each copy of $\text{Spec}(\mathbb{C})$ corresponding to 0 and ∞ . Finally pulling back through c_n yields $\text{Spec}(\mathbb{C}[x]/(x^n) \cup \text{Spec}(\mathbb{C}[x]/(x^n)))$.

(3) *The following calculation of fibers is more explicit. (Léo)*

Let the composition above be written $\varphi : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$, and the map of rings from which it is induced be $\tilde{\varphi} : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$ sending $x \mapsto (x^n - y^n)^2$ and $y \mapsto (x^n + y^n)^2$.

We now want to find the ramification points. Let $\mathfrak{p} = (\beta x - \alpha y)$ be fixed and non-the zero ideal. Recall that $\varphi^{-1}(V_+(\mathfrak{p})) = V_+(\tilde{\varphi}(\mathfrak{p}))^3$. The following commutative square below is a pullback

$$\begin{array}{ccc} V_+(\tilde{\varphi}(\mathfrak{p})) & \longrightarrow & \mathbb{P}_{\mathbb{C}}^1 \\ \downarrow & & \downarrow \varphi \\ V_+(\mathfrak{p}) & \longrightarrow & \mathbb{P}_{\mathbb{C}}^1 \end{array}$$

Moreover, one has that $V_+(\mathfrak{p}) \cong \text{Spec}(k(\mathfrak{p}))$. It is therefore enough to find out if $V_+(\tilde{\varphi}(\mathfrak{p}))$ is reduced to determine wether \mathfrak{p} is a ramification point or not, as one will have $(\mathbb{P}_{\mathbb{C}}^1)_{\bar{\mathfrak{p}}} \cong V_+(\tilde{\varphi}(\mathfrak{p}))$.

- $\alpha \neq \beta$: Moreover we assume that $\alpha, \beta \neq 0$. Without loss of generality, we may assume $\beta = 1$ and thus write $\mathfrak{p} = (x - \alpha y)$. We first show the following result.

Claim. *One has $V_+(\tilde{\varphi}(\mathfrak{p})) \subseteq D_+(x)$.*

³where $\tilde{\varphi}(\mathfrak{p})$ denotes the ideal generated by the image.

Proof. Let $\mathfrak{q} \in V_+(\tilde{\varphi}(\mathfrak{p})) \cap V_+(x)$, i.e one has

$$x, \tilde{\varphi}(x - \alpha y) = (1 - \alpha)(x^{2n} + y^{2n}) - 2(1 + \alpha)x^n y^n \in \mathfrak{q}.$$

But this implies that y^{2n} belongs to \mathfrak{q} and since the latter is prime, that $y \in \mathfrak{q}$. Since \mathfrak{q} does not contain the irrelevant ideal, we get a contradiction, and thus that the intersection must be empty. \square

Recall that

$$V_+(\tilde{\varphi}(\mathfrak{p})) \cong \text{Proj} \left(\mathbb{C}[x, y] / (\tilde{\varphi}(x - \alpha y)) \right)$$

as schemes. Since one also has $D_+(x) \cong \text{Spec} \left(\mathbb{C} \left[\frac{y}{x} \right] \right) \cong \text{Spec}(\mathbb{C}[t])$ with t sent to $\frac{y}{x}$, we get that

$$V_+(\tilde{\varphi}(\mathfrak{p})) \cong \text{Spec} \left(\mathbb{C}[t] / \left(t^{2n} - 2\frac{1+\alpha}{1-\alpha}t^n + 1 \right) \right).$$

Using the Chinese remainder theorem, one can rewrite

$$\mathbb{C}[t] / \left(t^{2n} - 2\frac{1+\alpha}{1-\alpha}t^n + 1 \right) = \mathbb{C}[t] / \left(t^n - \frac{(1+\sqrt{\alpha})^2}{1-\alpha} \right) \times \mathbb{C}[t] / \left(t^n - \frac{(1-\sqrt{\alpha})^2}{1-\alpha} \right)$$

which will be reduced, as one can use the Chinese Remainder theorem to get that both rings are isomorphic to a product of multiple copies of \mathbb{C} . It follows that $\mathfrak{p} = (\beta x - \alpha y)$ is not a ramification point.

- $\alpha = \beta$: We have $\tilde{\varphi}(x - y) = -4x^n y^n$. Once again, one has

$$V_+(x^n y^n) \cap V_+(x + y) = \emptyset$$

i.e $V_+(x^n y^n) \subseteq D_+(x + y)$. Let $t := \frac{x-y}{x+y}$. We have

$$D_+(x + y) \cong \text{Spec} \left(\mathbb{C}[x, y]_{(x+y)} \right) \cong \text{Spec}(\mathbb{C}[t]).$$

With this identification, one has

$$V_+(x^n y^n) \cong \text{Spec} \left(\mathbb{C}[t] / (1+t)^n(1-t)^n \right).$$

As one has

$$\mathbb{C}[t] / (1+t)^n(1-t)^n \cong \mathbb{C}[t] / (1+t)^n \times \mathbb{C}[t] / (1-t)^n$$

which is not reduced, we get that $(x - y)$ is a ramification point, whose fiber is

$$(\mathbb{P}_{\mathbb{C}}^1)_{(x-y)} \cong \text{Spec} \left(\mathbb{C}[t] / (1+t)^n \right) \bigsqcup \text{Spec} \left(\mathbb{C}[t] / (1-t)^n \right).$$

- $\alpha = 0$: We only do this case, as the case $\beta = 0$ is symmetric and will only result in a change of sign. Again, one has $V_+ \left((x^n - y^n)^2 \right) \cap V_+(x) = \emptyset$, we once again have

$$V_+ \left((x^n - y^n)^2 \right) \subseteq D_+(x).$$

This gives us

$$(\mathbb{P}_{\mathbb{C}}^1)_{\overline{(x)}} = V_+((x^n - y^n)^2) \cong \text{Spec}(\mathbb{C}[t]/(1 - t^n)^2)$$

and thus that (x) is a ramification point, as the affine scheme in the RHS is not reduced.

□