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EPFL, fall semester 2025
AG II - Schemes and sheaves

Solutions – week 3

Exercise 1. *Nilradical.* Let R be a ring. Denote by

$$\mathrm{nil}(R) := \{f \in R \mid f \text{ is nilpotent}\}.$$

(1) Show that
$$\mathrm{nil}(R) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}.$$
(2) Show that for an ideal $I \subset R$, we have $V(I) = \mathrm{Spec}(R)$ if and only if every element of I is nilpotent, meaning $I \subset \mathrm{nil}(R)$.

Solution key. Let $f \in R$ not nilpotent – let $S = f^{\mathbb{N}}$. Then $S^{-1}R \neq 0$. Therefore, there is a maximal ideal in $S^{-1}R$. This shows that there is a prime ideal not containing f.

□

Exercise 2. Spec *is an adjoint.* Let $(X, \mathcal{O}_X)$ be a scheme and A a ring. Show that the induced map on global sections

$$\mathrm{Hom}_{\mathrm{Sch}}((X, \mathcal{O}_X), \mathrm{Spec}(A)) \to \mathrm{Hom}_{\mathrm{Ring}}(A, \mathcal{O}_X(X))$$

is bijective. This implies that

$$\mathrm{Spec}\colon \mathrm{Ring}^{op} \to \mathrm{Sch}$$

is a right adjoint. In particular colimits of rings are sent to limits of schemes.

Solution key. We propose a proof assuming the anti-equivalence between rings and affine schemes.
It is straightforward to check the naturality of the map in X and A. We then just need to construct an inverse map to

$$\mathrm{Hom}_{\mathrm{Sch}}((X, \mathcal{O}_X), \mathrm{Spec}(A)) \to \mathrm{Hom}_{\mathrm{Ring}}(A, \mathcal{O}_X(X)).$$

We proceed in three steps. (See the document *Gluing arguments* for more precision on some points.)

(1) If X is affine, this is bijective because this is the statement of the anti-equivalence of categories between affine schemes and rings.
(2) Suppose then that X can be covered by affines
$$X = \bigcup_i U_i$$
such that their intersection is affine. We construct an inverse map. Let $\varphi\colon A \to \mathcal{O}_X(X)$ be a ring map. Denote by $\varphi_i\colon A \to \mathcal{O}_X(U_i)$ the composition of φ with the restriction. Using the anti-equivalence between rings and affine schemes (1), we get that φ_i correspond uniquely to a map of schemes $f_i\colon U_i \to \mathrm{Spec}(A)$. We want to show

that f_i and f_j coincide on $U_i \cap U_j$. As this intersection is affine by hypothesis we get that the restriction of f_i and f_j is the unique map of affine schemes which correspond to the map $\varphi_{ij} \colon A \to \mathcal{O}_X(U_{ij})$ which is φ composed by the restriction. Therefore we get a map of schemes $f \colon X \to \operatorname{Spec}(A)$. We check that it is the desired inverse. If $\varphi \colon A \to \mathcal{O}_X(X)$ is a ring map, the map on global sections induced by the above constructed f is φ by construction of the glued map. The other way around, if f is a map $X \to \operatorname{Spec}(A)$, we see by restricting to U_i that f is necessarily given by gluing of the maps induced by the above construction.

(3) We now consider X to be an arbitrary scheme. We want to construct an inverse map. We proceed exactly as above. The only difference is in the step when we want to compare f_i and f_j on $U_i \cap U_j$, which is not necessary affine. But $U_i \cap U_j$ is a scheme that can be covered with affine schemes such that their intersection is affine (see *Gluing arguments.*) Therefore we can use (2) to say that a map $U_i \cap U_j \to \operatorname{Spec}(A)$ is the same as a map of global sections $A \to \mathcal{O}_X(U_i \cap U_j)$. Therefore f_i and f_j are the same because they correspond to the the map $\varphi_{ij} \colon A \to \mathcal{O}_X(U_{ij})$ which is φ composed by the restriction as in the above case. Every other step goes similarly.

□

Remark. The above remains true if we replace Sch by the category of locally ringed spaces $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ring}}$. This characterizes Spec as the right adjoint of the global sections functor $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ring}} \to \mathrm{Ring}^{op}$. This formalize the saying that $\operatorname{Spec}(R)$ is the universal (locally ringed) space such that R is the ring of global functions on this space.

Exercise 3. *Reduced schemes.* A scheme $(X, \mathcal{O}_X)$ is *reduced* if for all opens U of X the ring $\mathcal{O}_X(U)$ is reduced.

(1) Show that a scheme $(X, \mathcal{O}_X)$ is reduced if and only if for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a reduced ring.
(2) Show that an affine scheme $\operatorname{Spec}(A)$ is reduced if and only if A is a reduced ring.

The *reduction* of a scheme X is a scheme X_{red} together with a map $\iota : X_{red} \to X$ with the property that for every map $Y \to X$ where Y is a reduced scheme, then Y factors uniquely to ι.

(3) Show that if $X = \operatorname{Spec}(A)$ then $\operatorname{Spec}(A/\operatorname{nil}(A)) \to \operatorname{Spec}(A)$ is the reduction of $\operatorname{Spec}(A)$.
(4) Show that the reduction of any scheme exists and that $\iota \colon X_{red} \to X$ is a homeomorphism.

Solution key. (1) Suppose that $(X, \mathcal{O}_X)$ is reduced. Take $s_x \in \mathcal{O}_{X,x}$ such that $s_x^n = 0$. First take an U where s_x lifts to a section $s \in \mathcal{O}_X(U)$. Then s^n is sent to 0 in $\mathcal{O}_{X,x}$. It implies that there is a smaller open V such that $s^n = 0$. But as $\mathcal{O}_X(V)$ is reduced, we deduce that $s = 0$ in $\mathcal{O}_X(V)$ proving that $s_x = 0$ as wanted.

For the other direction, take $f \in \mathcal{O}_X(U)$ nilpotent. Then every image in all stalks for all $x \in U$ are nilpotent implying that $f_x = 0$ for all $x \in U$ and then $f = 0$.

(2) If $\mathrm{Spec}(A)$ is reduced then taking global sections we deduce that A is reduced as a ring.

For the other way around, we prove the following:

Claim. *If S is any multiplicative subset of A and A is reduced, then $S^{-1}A$ is also reduced.*

Indeed, if $\frac{a^n}{s^n} = 0$, it means that there is some N and $s' \in S$ such that $s'^N a^n = 0$. But then, we see that $s'a$ is nilpotent, of order at most $M = \max\{N, n\}$. As A is reduced, $s'a = 0$ implying that a is mapped to zero in $S^{-1}A$.

Therefore for every prime $\mathfrak{p}$ of A, $A_\mathfrak{p}$ is reduced, showing that $\mathrm{Spec}(A)$ is reduced.

(3) We show that $\mathrm{Spec}(A_{red}) \to \mathrm{Spec}(A)$ is the reduction in the category of schemes. Let $Y \to \mathrm{Spec}(A)$ a map, where Y is a reduced scheme. By adjunction, this is the same as the data of a map $A \to \mathcal{O}_Y(Y)$. Because the target is reduced, this map factors uniquely to $A \to A_{red}$. By adjunction again, we get the unique desired map $Y \to \mathrm{Spec}(A_{red})$.

(4) Define a scheme X_{red} with the same underlying topological space, but with $\mathcal{O}_{X_{red}}$ being the sheafification of $U \to \mathcal{O}_X(U)_{red}$. Let (U_i) be a basis of X consisting only of affine open sub-schemes. For every open affine $U_i = \mathrm{Spec}(A_i)$ the presheaf define above is equal to $\mathcal{O}_{\mathrm{Spec}(A_{i,red})}$ on open affines of U_i. Therefore, because this presheaf already defines a sheaf on a basis of open subsets, this implies that the sheafification equals it on these affine opens (but not necessarily on other opens). Therefore we conclude that $\mathcal{O}_{\mathrm{Spec}(A_{i,red})}$ is equal to the sheafification of the presheaf defined above on $\mathrm{Spec}(A_i)$. It follows that X_{red} with the same toplogical space as X and the sheaf define above is a scheme.

Now for the universal property, if $f : Y \to X$ is a morphism with Y reduced, then topologically there is evidently a unique lift $Y \to X_{red}$. For the sheaf part consider the map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y.$$

Because $f_*\mathcal{O}_Y$ is reduced on every open this factors uniquely through the presheaf reduction and then to the sheafification $\mathcal{O}_{X,red}$ by universal property of the sheafification. This is what we wanted.

□

Exercise 4. *Residue fields and rational points.* Let $(X, \mathcal{O}_X)$ be a scheme, $x \in X$ and $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ the *residue field at x*.

(1) Let K be a field. Show that a map $\mathrm{Spec}(K) \to X$ with topological image x amounts to a field extension $k(x) \to K$.
(2) Let k be a field. Fix $X \to \mathrm{Spec}(k)$ a map for the rest of the exercise. Show that for all $x \in X$, $k(x)$ is naturally a field extension of k.

(3) We say that $x \in X$ is *k-rational* if the natural extension of last item $k \to k(x)$ is an isomorphism. Show that the set of k-rational points of X is identified with the set of maps $\mathrm{Spec}(k) \to X$ such that the composite $\mathrm{Spec}(k) \to X \to \mathrm{Spec}(k)$ is the identity.

(4) Let now $X = \mathrm{Spec}(k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)) \to \mathrm{Spec}(k)^1$, where $f_1, \ldots, f_m$ are polynomials. Show that the set of k-rational points of X is identified with the set of solutions in k^n of the system of polynomials $f_1, \ldots, f_m$.

Solution key. (1) At the topological level, a map $f\colon \mathrm{Spec}(K) \to X$ is just the data of a point $x \in X$, the image of this map. There is also the data of a map of sheaves of rings

$$\varphi\colon \mathcal{O}_X \to f_*K$$

But note that f_*K is the skyscraper sheaf of K at x. Then we have a factorization

$$\varphi_x\colon \mathcal{O}_{X,x} \to K.$$

Because the map is locally ringed, we have that $\varphi_x^{-1}(0) = \mathfrak{m}_x$. Therefore we see that we get a map $k(x) \to K$. Given such a map, one can see that one can define a scheme as explained above.

(2) Simply comes from the composition

$$\mathrm{Spec}(k(x)) \to X \to \mathrm{Spec}(k)$$

(3) If the composition $c\colon \mathrm{Spec}(k(x)) \to \mathrm{Spec}(k)$ is an isomorphism, then $c^{-1}\colon \mathrm{Spec}(k) \to \mathrm{Spec}(k(x)) \to X \to \mathrm{Spec}(k)$ is the identity. One sees that this construction is bijective.

(4) This is because they correspond to k-algebra surjections

$$k[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \to k$$

where the claim is now the universal property of the quotient.

□

Exercise 5. *Exceptional functors (1).* Let X be a topological space. Let $j\colon U \to X$ be an open subset and $\iota\colon Z \to X$ its closed complement. We work with categories of sheaves of abelian groups on these spaces.

(1) Consider $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(Z)$. Compute every stalk of $\iota_*\mathcal{F}$.
(2) Show that ι_* is exact.
(3) Give an example to show that j_* is not exact.

Consider $\mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(U)$. We define the *extension by zero* or *exceptional direct image* $j_!\mathcal{G}$ to be the sheafification of the presheaf defined by $V \mapsto \mathcal{G}(V)$ if $V \subset U$ and 0 otherwise.

(4) Show that for every sheaf $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$ there is a natural exact sequence

$$0 \to j_!j^{-1}\mathcal{H} \to \mathcal{H} \to \iota_*\iota^{-1}\mathcal{H} \to 0.$$

[1] Induced by the inclusion $k \to k[x_1, \ldots, x_n]$

(5) Show that there is a natural bijection in $\mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(U)$ and $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(U)}(\mathcal{G}, j^{-1}\mathcal{H}) \cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X)}(j_!\mathcal{G}, \mathcal{H}).$$

This means that for an open immersion j, we have a sequence of adjoints $j_! \dashv j^{-1} \dashv j_*$.

Solution key. (1) If $x \in Z$ then we see that we have a natural isomorphism

$$(\iota_*\mathcal{F})_x \cong \mathcal{F}_x.$$

If $x \notin Z$ then as $\iota_*\mathcal{F}(X \setminus Z) = \mathcal{F}(\emptyset) = 0$ we see that $(\iota\mathcal{F})_x = 0$.

(2) To check the exactness of a sequence, we check it at stalks. Therefore the exactness of ι_* follows from the previous computation.

(3) Consider $U = \mathbb{C} \setminus 0 \to \mathbb{C}$. Consider the exponential sequence ($\mathcal{O}$ denotes sheaves of holomorphic functions and $\mathcal{O}^\times$ the sheaf of non-vanishing holomorphic functions)

$$1 \to \mathbb{Z} \to \mathcal{O}_U \xrightarrow{\exp} \mathcal{O}_U^\times \to 1.$$

We claim that $j_*\mathcal{O}_U \to j_*\mathcal{O}_U^\times$ is not surjective. By contradiction, if it is, it would be surjective at the stalk at zero

$$(j_*\mathcal{O}_U)_0 \to (j_*\mathcal{O}_U^\times)_0.$$

In particular the germ of the inclusion map $g\colon U \to \mathbb{C} \setminus 0$ would be attained by some element. This means that there exists $V \subset U$ with $f \in \mathcal{O}(V)$ with $\exp(f) = g$. This a contradiction, for example to Cauchy formula.

(4) First, remark that if $\mathcal{G} \in \mathrm{Sh}(U)$, then stalks of $\iota_!\mathcal{G}$ behave the following way. If $x \in U$ we have a natural isomorphism

$$(\iota_!\mathcal{G})_x \to \mathcal{G}_x,$$

and if $x \notin U$ we have $(\iota_!\mathcal{G})_x = 0$. The exactness follows from the computation at stalks. If $x \in U$ then it amounts to an isomorphism and then the zero map, and if $x \in Z$ first the zero map, and then an isomorphism.

(5) First note that

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X)}(j_!\mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}_{\mathrm{PSh}_{\mathrm{Ab}}(X)}(j_!^{pr}\mathcal{G}, \mathcal{H}).$$

Where $j_!^{pr}$ denotes the extension by zero before sheafification. We see that a morphism $j_!^{pr}\mathcal{G} \to \mathcal{H}$ amounts to a morphism $\mathcal{G} \to j^{-1}\mathcal{H}$. Indeed if $V \not\subset U$ we have $j_!^{pr}\mathcal{G}(V) = 0$. So a map $j_!^{pr}\mathcal{G} \to \mathcal{H}$ just amounts to maps $\mathcal{G}(V) \to \mathcal{H}(V)$ which are compatible with restrictions for every $V \subset U$. In other words this exactly the data of a map of sheaves $\mathcal{G} \to j^{-1}\mathcal{H}$. This association is natural and bijective.

□

Exercise 6. *Exceptional functors (2).* We keep setup and notation as in previous exercise. Let $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$.

(1) Show that for every $s \in \mathcal{H}(V)$ for an open V, then
$$\mathrm{supp}(s) := \{x \in V \mid s_x \neq 0\}$$
is closed.

(2) Show that $\mathcal{H}_Z$, the presheaf on X defined by
$$\mathcal{H}_Z(V) = \{s \in \mathcal{H}(V) \mid \mathrm{supp}(s) \subset Z \cap V\}$$
is a sheaf. Show that $\mathcal{H}_Z(V)$ is the kernel of the map
$$\mathcal{H}(V) \to \mathcal{H}(V \cap (X \setminus Z)).$$

(3) Show that if $V' \subset V$ such that $V' \cap Z = V \cap Z$ then the restriction map $\mathcal{H}_Z(V) \to \mathcal{H}_Z(V')$ is an isomorphism.

(4) Show that for any sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(Z)$ any map $\iota_*\mathcal{F} \to \mathcal{H}$ factors through $\mathcal{H}_Z$.

We define the *exceptional inverse image* $\iota^!\mathcal{H} := \iota^{-1}\mathcal{H}_Z$.

(5) Show that there is a natural bijection in $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(Z)$ and $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$
$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(Z)}(\mathcal{F}, \iota^!\mathcal{H}) \cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X)}(\iota_*\mathcal{F}, \mathcal{H}).$$
This means that for a closed immersion ι, we have a sequence of adjoints $\iota^{-1} \dashv \iota_* \dashv \iota^!$.

Solution key. (1) If $x \in X$ is such that $s_x = 0$ there is an open set around x with $s = 0$ in this open set.

(2) We show that $\mathcal{H}_Z$ is the kernel map of the unit map (so the fact that it is a sheaf will follow from this description)
$$\mathcal{H} \to j_*j^{-1}\mathcal{H}$$
which is on each open set V the restriction
$$\mathcal{H}(V) \to \mathcal{H}(V \cap U).$$
But elements s which are sent to zero by this restriction are exactly elements such that $s_x = 0$ for all $x \in V \cap U$. This happens if and only if that $\mathrm{supp}(s) \subset Z \cap V$.

(3) Let $V' \subset V$ with $V' \cap Z = V \cap Z$. It implies that
$$V = V' \cup (V \cap (X \setminus Z)).$$
We show that
$$\mathcal{H}_Z(V) \to \mathcal{H}_Z(V')$$
is an isomorphism. We show the injectivity. if s is sent to zero, then note that $s_{V'} = 0$ and $s_{V \cap (X \setminus Z)} = 0$ by construction. So $s = 0$. The surjectivity follows from gluing. If $s \in \mathcal{H}_Z(V')$ we can glue $s \in \mathcal{H}(V)$ and $0 \in \mathcal{H}(V \cap (X \setminus Z))$ to get a section of $\mathcal{H}_Z(V)$.

(4) Follows by computation at stalks at $x \in U$.

(5) Note that the presheaf preimage of $\mathcal{H}_Z$ can be expressed as, on an open set $W \subset Z$ of Z, by (the colimit ranges over opens V of X such that $V \cap Z = W$)
$$\varinjlim_{\substack{V \subset X \\ V \cap Z = W}} \mathcal{H}_Z(V).$$

Note that therefore by point (3) above this colimit is taken on isomorphisms: we mean by this that every morphism in the diagram is an isomorphism. This implies that the colimit is equal to the limit on the same system. With the fact that this colimit is taken on isomorphism we also see that this presheaf is already a sheaf.

Note first that

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X)}(\iota_*\mathcal{F}, \mathcal{H}) = \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X)}(\iota_*\mathcal{F}, \mathcal{H}_Z)$$

by point (4). Let W be an open of Z. A morphism $\mathcal{F} \to \iota^!\mathcal{H}$ on W amounts to a collection of morphisms $\mathcal{F}(W) \to \mathcal{H}_Z(V)$ for every $V \subset X$ open with $V \cap Z = W$ that commutes with restrictions (the colimit equals the limit). Therefore a map of sheaves $\mathcal{F} \to \iota^!\mathcal{H}$ amounts to a map for every open set $U \subset X$ of X from $\mathcal{F}(U \cap Z) \to \mathcal{H}_Z(U)$ which is compatible with every restriction. In other words, this is the data of morphism of sheaves $\iota_*\mathcal{F} \to \mathcal{H}_Z$. These identifications are natural and bijective.

□

Exercise 7. *Topological properties of schemes.* A topological space X is T_0 if for every pair of different elements $x, y \in X$ there exist an open set U of X such that exactly x or y is in U.

(1) Let X be the underlying topological space of a scheme. Show that X is T_0.

A topological space is called *irreducible* if it cannot be written as the union of two proper and non-empty closed subsets.

(1) Show that any non-empty open set of an irreducible topological space is dense.
(2) Show that if an irreducible topological space X contains at least two points, then X is not Hausdorff.
(3) Let A be a ring. Show that the topological space $\mathrm{Spec}(A)$ is irreducible if and only if A_{red} is an integral domain.

A topological space is called *sober* if for any non-empty irreducible closed subset $Z \subset X$, there exist a unique point $\eta_Z \in Z$ such that $\overline{\{\eta_Z\}} = Z$. In this case, we call η_Z the *generic point* of Z.

(1) Show that any Hausdorff topological space is sober.
(2) Let X be the underlying topological space of a scheme. Show that X is sober.
(3) Let A be an integral domain. What is the generic point of $\mathrm{Spec}(A)$?