

Solutions – week 2

Exercise 1. *Sheaves of abelian groups.* Let X be a topological space. Let \mathcal{F}, \mathcal{G} be sheaves of abelian groups and $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ a morphism of sheaves of abelian groups.

- (1) Let $\ker(\varphi)$ and $\text{im}(\varphi)$ be respectively the kernel sheaf and the image sheaf.¹ Show that for every $x \in X$, one can define natural maps which are isomorphisms

$$\ker(\varphi)_x \rightarrow \ker(\varphi_x) \text{ and } \text{im}(\varphi)_x \rightarrow \text{im}(\varphi_x).$$

- (2) Show that φ is an injective morphism of sheaves (*i.e.* injective on every open) if and only if for every $x \in X$ the morphism of abelian groups $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is injective
- (3) Show that φ the following condition are equivalent.
- (a) For every $x \in X$ the map $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is surjective.
 - (b) for every U open and $s \in \mathcal{G}(U)$, there exists an open cover $U = \bigcup U_i$ and sections $t_i \in \mathcal{F}(U_i)$ with $\varphi(t_i) = s|_{U_i}$.

In this case we say that the map is an *epimorphism of sheaves* or a *surjective map of sheaves*. This is not to be mistaken with a surjective map of presheaves: the latter simply means that we have surjection on each open and this is strictly stronger than being an epimorphism of sheaves. See (6).

- (4) Show that the natural map $\text{im}(\varphi) \rightarrow \mathcal{G}$ is injective.
- (5) Show that φ is an isomorphism if and only if it is an injective morphism of sheaves and a surjective morphism of sheaves.
- (6) Let $f = X \rightarrow *$ be the unique morphism to the point. Show that $f_* = \Gamma(X, -) : \text{Sh}_{\text{Ab}}(X) \rightarrow \text{Ab}$ is left-exact. Give an example to show that f_* is not right-exact in general.

Exercise 2. *Gluing sheaves.* Let X be a topological space and $\bigcup U_i = X$ an open cover of X . Let $(\mathcal{F}_i \in \text{Sh}(U_i), \varphi_{ij})$ be a collection of sheaves on $\text{Sh}(U_i)$ together with isomorphisms

$$\varphi_{ij} : \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}}$$

in $\text{Sh}(U_{ij})$ satisfying for each i that $\text{id} = \varphi_{ii}$ and for each i, j, k the following *cocycle condition* $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$.

Show that there exists a unique² sheaf $\mathcal{F} \in \text{Sh}(X)$ with maps $\psi_i : \mathcal{F}|_{U_i} \rightarrow \mathcal{F}_i$ with the following universal property: for all sheaves $\mathcal{G} \in \text{Sh}(X)$ we have a

¹The kernel sheaf is the kernel presheaf but the image sheaf is the *sheafification* of the image presheaf.

²up to isomorphism.

a bijection

$$\mathrm{Hom}(\mathcal{G}, \mathcal{F}) \cong \left\{ (\mathcal{G}|_{U_i} \xrightarrow{f_i} \mathcal{F}_i) \in \prod_i \mathrm{Hom}(\mathcal{G}|_{U_i}, \mathcal{F}_i) \mid \text{s.t. for all } i, j : \varphi_{ij} f_i = f_j \right\}$$

given by $f \mapsto \psi_i f|_{U_i}$.

Show furthermore that ψ_i are isomorphisms.

Solution key. Let $\iota_i : U_i \rightarrow X$ for the inclusion of the open set.

Some quick remarks : using cocycle condition, we get $\varphi_{ii} = \varphi_{ii} \circ \varphi_{ii}$. By hypothesis φ_{ii} are isomorphisms so we get : $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$. Then using the cocycle condition : $\mathrm{id}_{\mathcal{F}_i} = \varphi_{ji} \circ \varphi_{ij}$ and $\mathrm{id}_{\mathcal{F}_j} = \varphi_{ij} \circ \varphi_{ji}$.

We define ³ \mathcal{F} on an open set U by :

$$\mathcal{F}(U) = \left\{ (s_i) \in \prod_i \mathcal{F}_i(U \cap U_i) \mid \forall (i, j) \quad s_j|_{U_{ij}} = \varphi_{ij}(s_i|_{U_{ij}}) \right\}$$

as *sub-(pre)sheaf of the product sheaf* $\prod_i \iota_{i*} \mathcal{F}_i$. If $V \subset U$ note that the restriction $(s_i) \mapsto (s_i|_V)$ is well defined because : $s_j|_{U_{ij} \cap V} = \varphi_{ij}(s_i|_{U_{ij}})|_V = \varphi_{ij}(s_i|_{U_{ij} \cap V})$, using the fact that φ_{ij} is a morphism of sheaves.

- We show that \mathcal{F} is indeed a sheaf. Let $V = \cup_\alpha V_\alpha$ an open cover. Let $((s_i^\alpha)_i)_\alpha$ be a collection of elements lying in $\mathcal{F}(V_\alpha)$, such that we have $s_i^\alpha|_{V_\alpha} = s_i^\beta|_{V_\beta}$ for any α, β . Using the sheaf property on the product sheaf (which follows directly from the sheaf property of each factor), we get a *unique* element $(s_i) \in \prod_i \iota_{i*} \mathcal{F}_i(V)$ lifting the collection. We show that this unique element lies in fact in $\mathcal{F}(V)$. We need to show that for any i, j we have $s_j|_{U_{ij}} = \varphi_{ij}(s_i|_{U_{ij}})$. But when we restrict both sides of the desired equality on V_α , the equality holds because $(s_i^\alpha)_i$ lies in $\mathcal{F}(V_\alpha)$. So using again the uniqueness in the sheaf property of the product sheaf, we get what we want.
- To show that \mathcal{F} is unique up to isomorphism in $\mathrm{Sh}(X)$ we spell out an universal property that it verifies. We write $(\mathcal{F} \xrightarrow{p_i} \iota_{i*} \mathcal{F}_i)_i$ the collection of sheaf morphisms induced by the projections from the product. We claim that \mathcal{F} satisfies the following universal property :

For all $\mathcal{G} \in \mathrm{Sh}(X)$ and collections $(\mathcal{G} \xrightarrow{f_i} \iota_{i*} \mathcal{F}_i)_i$ of sheaf morphisms such that :

for all U open and $\forall t \in \mathcal{G}(U)$, we have for all i, j :

$$f_j(t)|_{U_{ij}} = \varphi_{ij}(f_i(t)|_{U_{ij}})$$

there is a unique sheaf morphism $\mathcal{G} \xrightarrow{f} \mathcal{F}$ such that for all i ,

$$p_i f = f_i.$$

This is indeed the case : if we take a collection $(\mathcal{G} \xrightarrow{f_i} \iota_{i*} \mathcal{F}_i)_i$, we get a map f from \mathcal{G} to the product $\prod_i \iota_{i*} \mathcal{F}_i$ by the universal property

³One should question the coherence of this definition : let $(s_i) \in \mathcal{F}(U)$. Then

$$(1) \quad s_j|_{U_{ij}} = \varphi_{ij}(s_i|_{U_{ij}}) = \varphi_{ij}(\varphi_{ji}(s_j|_{U_{ij}})) = s_j|_{U_{ij}}$$

using the property for (i, j) and (j, i) and $\mathrm{id}_{\mathcal{F}_j} = \varphi_{ij} \circ \varphi_{ji}$. So (1) highlight why the fact that φ_{ij} and φ_{ji} are inverses to each other is important in this gluing process.

of the product in $\text{Sh}(X)$. But the condition $f_j(t)|_{U_{ij}} = \varphi_{ij}(f_i(t)|_{U_{ij}})$ for all i, j says exactly that in fact f factors into \mathcal{F} .

- Now we show that $\varphi_k : \mathcal{F}_{U_k} \rightarrow \mathcal{F}_k$ induced by the projection is an isomorphism of sheaves for all k . To show surjectivity we will use crucially the cocycle condition.

- (1) Surjectivity. Let $V \subset U_k$ open. Let $s_k \in \mathcal{F}_k(V)$. We want to construct an element $(s_i) \in \mathcal{F}(V) \subset \prod_i \mathcal{F}_i(V \cap U_i)$ such that its k -th component is s_k .

For each i , we define s_i using the cover $V \cap U_i = \cup_j V \cap U_{ij}$, and the collection $(\varphi_{ki}(s_k|_{U_{ij}}))$ of elements in $\mathcal{F}_i(V \cap U_{ij})$. It verifies the intersection property because φ_{ij} is a morphism of sheaves. So s_i is defined by $s_i|_{U_{ij}} = \varphi_{ki}(s_k|_{U_{ij}})$. Note that if $i = k$ the element defined in this way is s_k , because φ_{kk} is the identity.

Now we claim that the collection (s_i) that we just defined is indeed in $\mathcal{F}(V)$. To show this, we need to show that for any i, j , we have : $s_j|_{U_{ij}} = \varphi_{ij}(s_i|_{U_{ij}})$. But :

$$s_j|_{U_{ij}} = \varphi_{kj}(s_k|_{U_{ij}}) = \varphi_{ij} \circ \varphi_{ki}(s_k|_{U_{ij}}) = \varphi_{ij}(\varphi_{ki}(s_k|_{U_{ij}})) = \varphi_{ij}(s_i|_{U_{ij}})$$

Using the definition of s_i 's and the cocycle condition.

- (2) Injectivity. Let (s_i) and (s'_i) be two elements in $\mathcal{F}(V)$ such that their k -th component is $s_k = s'_k$. Now one gets for any i :

$$s_i = \varphi_{ik}(s_k) = \varphi_{ik}(s'_k) = s'_i$$

thus proving the injectivity.

Remark. One can interpret the result of the previous exercise as saying that the presheaf with values in categories

$$\text{Sh} : \text{Ouv}(X)^{\text{op}} \rightarrow \text{Cat}$$

is a sheaf in a suitable sense. □

Remark. Can you see how the last exercise resembles the following statement: “ $U \mapsto \text{Sh}(U)$ is a sheaf”?

Exercise 3. *Inverse image.* Let $f : X \rightarrow Y$. Let $\mathcal{F} \in \text{Sh}(Y)$. We define the presheaf on X

$$f^\# \mathcal{F}(U) = \varinjlim_{V \supset f(U)} \mathcal{F}(V).$$

- (1) Show that if $f : * \rightarrow X$ is a point $x \in X$ then $f^\# \mathcal{F} = \mathcal{F}_x$.
- (2) Show that if $y = f(x)$ then there is a natural isomorphism

$$(f^\# \mathcal{F})_x \rightarrow \mathcal{F}_y.$$

- (3) Show that if f is an open immersion, then $f^\#$ is a sheaf.
- (4) Find an example of map of topological spaces $f : X \rightarrow Y$ and a sheaf \mathcal{F} on Y such that $f^\# \mathcal{F}$ is *not* a sheaf.

- (5) Let $f^{-1}\mathcal{F}$ be the sheafification of $f^{\#}\mathcal{F}$. We call this sheaf the *inverse image* of \mathcal{F} . Show that the $f^{-1} \dashv f_*$ ⁴ meaning that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{F}, f_*\mathcal{G}).$$

Exercise 4. Localization Let R be a ring. Let S be a multiplicative subset.

- (1) Describe the points of $\mathrm{Spec}(S^{-1}R)$. If $\mathfrak{p} \in \mathrm{Spec}(R)$ show that $\mathrm{Spec}(R_{\mathfrak{p}})$ is the intersection of all opens containing \mathfrak{p} .
 (2) Let M be an R -module and $I \supseteq R$ an ideal. Show that there is an isomorphism

$$S^{-1}(M/I) \cong (S^{-1}M)/(IS^{-1}M).$$

- (3) Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and $I \subseteq R$ an ideal. When

$$(R/I)_{\mathfrak{p}} = 0 \quad ?$$

Can you interpret this geometrically?

- (4) Let R be integral. Identify the image of the injective map $S^{-1}R \rightarrow \mathrm{Frac}(R)$.

Exercise 5. Affine schemes are quasi-compact. Let R be a ring. Show that $\mathrm{Spec}(R)$ is quasi-compact.⁵ Deduce that the underlying topological space of any (affine) scheme has a basis of quasi-compact open subsets.

Solution key. Let R be a ring. Let (a_i) be a collection of elements such that

$$\mathrm{Spec}(R) = \bigcup_i D(a_i).$$

This means that $1 \in (a_i)$. Therefore there exists a_1, \dots, a_n and $b_1, \dots, b_n \in R$ such that

$$1 = \sum_{j=1}^n b_j a_j$$

for some n . Therefore

$$\mathrm{Spec}(R) = \bigcup_{j=1}^n D(a_j).$$

□

Exercise 6. Connected affine schemes. We say that a ring R is *connected* if for all $a, b \in R$ if

$$a + b = 1 \text{ and } ab = 0$$

then exactly one of the two elements is non-zero.

- (1) Show that it is equivalent to the fact there is exactly two idempotents (namely 0 and 1) in the ring R .
 (2) Show that R is connected if and only if $\mathrm{Spec}(R)$ is connected.

⁴We say that f^{-1} is *left adjoint* to f_*

⁵A topological space X is *quasi-compact* if every open cover of X can be refined to a finite cover.

Solution key. For (1) one sees that the correspondence between the two statements is given by the fact that if $a + b = 1$, and $ab = 0$ then a and b are idempotents – in the other direction if e is idempotent, note that $a = e$ and $1 - e = b$ satisfy the above.

For (2) if $\text{Spec}(R)$ is not connected, then $U \sqcup V = \text{Spec}(R)$ for non empty opens U and V . But then using the sheaf property of the structure sheaf of $\text{Spec}(R)$

$$R = \mathcal{O}_U(U) \times \mathcal{O}_V(V)$$

where both $\mathcal{O}_U(U)$ and $\mathcal{O}_V(V)$ are non-zero rings because the opens are not empty. Therefore, we see that there is non-trivial idempotents. The other directions is given by considering $D(e)$ and $D(1 - e)$ for a non-trivial idempotent e .

Remark. Let I be an ideal such that $V(I)$ is open and closed. We claim that $I = (e)$ for an idempotent e . Indeed, $V(I)$ is an affine scheme so $\mathcal{O}_{V(I)}(V(I)) = R/I$. The complement is also closed, say of the form $V(J)$ – it is also affine so $\mathcal{O}_{V(J)}(V(J)) = R/J$. But now by assumption

$$\text{Spec}(R) = \text{Spec}(R/I) \times \text{Spec}(R/J)$$

where both schemes in the decomposition are supposed to be open subschemes. By the sheaf property of the structure sheaf, we get

$$R \cong R/I \times R/J$$

and we see that I is generated by the preimage of $(0, 1)$.

Exercise 7. Stalks, morphisms and cotangent spaces

- (1) Let $X \rightarrow Y$ be a continuous map between topological spaces, and \mathcal{F} a sheaf on X . Let $x \in X$ and $y = f(x)$. Show that there is a natural map

$$(f_*\mathcal{F})_y \rightarrow \mathcal{F}_x.$$

Remark. This is used to define the *induced map on local rings* of a map of locally ringed spaces. Namely if $(f, f^\sharp): (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is one, with $f^\sharp: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$, the induced map on local rings is for $x = f(y)$,

$$\mathcal{O}_{Y,y} \xrightarrow{f_y^\sharp} (f_*\mathcal{O}_X)_y \rightarrow \mathcal{O}_{X,x}.$$

- (2) Let R be an integral domain. Consider $\varphi: R[x, y] \rightarrow R[x, y]$ defined by $x \mapsto xy$ and $y \mapsto y$. Consider

$$f: \text{Spec}(R[x, y]) \rightarrow \text{Spec}(R[x, y])$$

the induced map on associated affine schemes.⁶ Show that for all $\lambda \in R$ we have $f((x - \lambda, y)) = (x, y)$.

- (3) Let now $R = k$ a field. With point (1) and the remark there is induced map on local rings

$$k[x, y]_{(x,y)} \rightarrow k[x, y]_{(x-\lambda,y)}.$$

⁶Recall that the induced map on Spec is given by the *preimage* φ^{-1}

We write $\mathfrak{m}_{(0,0)} := \mathfrak{m}_{(x,y)}$ and $\mathfrak{m}_{(\lambda,0)} := \mathfrak{m}_{(x-\lambda,y)}$ for the maximal ideals of these local rings. Understand the induced k -linear map

$$\mathfrak{m}_{(0,0)}/\mathfrak{m}_{(0,0)}^2 \rightarrow \mathfrak{m}_{(\lambda,0)}/\mathfrak{m}_{(\lambda,0)}^2.$$

This means the following: find a k -basis of these vector spaces and describe the matrix of the map in terms of your chosen basis.

Remark. We will later see that these vector spaces are the *cotangent spaces* at $(0,0)$ and $(\lambda,0)$ respectively and that the map that you studied is *the precomposition by the differential of f at these points*.

Solution key. This exercise was a previous hand in exercise, and solutions are credited to past students of the course.

(2)(Alissa) Let R be an integral domain. Consider $\phi : R[x, y] \rightarrow R[x, y]$ a ring homomorphism such that $x \mapsto xy$ and $y \mapsto y$. Consider now the map $f : \text{Spec}(R[x, y]) \rightarrow \text{Spec}(R[x, y])$ induced by the map ϕ . We show that for every $\lambda \in R$ we have that $f((x - \lambda, y)) = \phi^{-1}(x - \lambda, y) = (x, y)$. To prove this point, consider the following commutative diagram

$$\begin{array}{ccc} R[x, y] & \xrightarrow{\phi} & R[x, y] \\ & \searrow \text{ev}_{(0,0)} & \swarrow \text{ev}_{(\lambda,0)} \\ & & R \end{array}$$

We have that $\text{ev}_{(\lambda,0)} \circ \phi = \text{ev}_{(0,0)}$. Hence we have the following series of equalities

$$\begin{aligned} (x, y) &= \ker(\text{ev}_{(0,0)}) = \text{ev}_{(0,0)}^{-1}(0) = (\text{ev}_{(\lambda,0)} \circ \phi)^{-1}(0) \\ &= \phi^{-1}(\text{ev}_{(\lambda,0)}^{-1}(0)) = \phi^{-1}(x - \lambda, y) = f(x - \lambda, y) \end{aligned}$$

This proves our point.

(3)(Maxence) Now let $R = k$ be a field. We have a local homomorphism of local rings

$$f_{(x,y)}^\# : \mathcal{O}_{\text{Spec}(k[x,y]),(x,y)} \rightarrow \mathcal{O}_{\text{Spec}(k[x,y]),(x-\lambda,y)}.$$

But we know that $\mathcal{O}_{\text{Spec}(k[x,y]),\mathfrak{p}} \cong k[x, y]_{\mathfrak{p}}$ for any $\mathfrak{p} \in \text{Spec}(k[x, y])$.

Thus we can see $f_{(x,y)}^\#$ as a local homomorphism of local rings $k[x, y]_{(x,y)} \rightarrow k[x, y]_{(x-\lambda,y)}$. Set $\mathfrak{m}_{(0,0)}$ and $\mathfrak{m}_{(\lambda,0)}$ to be the maximal ideal of respectively $k[x, y]_{(x,y)}$ and $k[x, y]_{(x-\lambda,y)}$.

We want to understand the k -linear map $\mathfrak{m}_{(0,0)}/\mathfrak{m}_{(0,0)}^2 \rightarrow \mathfrak{m}_{(\lambda,0)}/\mathfrak{m}_{(\lambda,0)}^2$.

For any maximal ideal \mathfrak{m} of $k[x, y]$, we have the following isomorphism of k -vector spaces ($k[x, y]/\mathfrak{m}$ -vector spaces) :

$$\mathfrak{m}_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}^2 \cong \mathfrak{m}/\mathfrak{m}^2$$

where $\mathfrak{m}_{\mathfrak{m}}$ is the maximal ideal of $k[x, y]_{\mathfrak{m}}$.

Furthermore, it is easy to see that $\{\bar{x}, \bar{y}\}$ is a k -basis of $(x, y)/((x^2, xy, y^2))$ and $\{\overline{x-\lambda}, \bar{y}\}$ is a k -basis of $(x - \lambda, y)/((x - \lambda)^2, (x - \lambda)y, y^2)$ since they are k -linear independent elements in their respective quotient. Since the induced linear map is just defined by applying φ , we get that $\varphi(\bar{x}) = \overline{xy} = \lambda\bar{y}$ and $\varphi(\bar{y}) = \bar{y}$ by definition of elements in the quotient $(x - \lambda, y)/((x - \lambda)^2, (x - \lambda)y, y^2)$.

That is, by taking bases as above, the linear map that we are looking for can be describe as the following matrix

$$\begin{pmatrix} 0 & 0 \\ \lambda & 1 \end{pmatrix}.$$

□