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AG II - Schemes and sheaves

Solutions – week 1

Exercise 1. *Refresh*
The goal of this exercise is to refresh some notions of commutative algebra as well as their interpretation in algebraic geometry. Let k be an algebraically closed field.

(1) Recall (your definition) of $\mathbb{A}^n_k$ and that the polynomnial algebra $k[x_1, \ldots, x_n]$ is to be interpreted as *functions* on this space.
(2) A *finite type k-algebra A* is a k-algebra who admits a surjection $f : k[x_1 \ldots, x_n] \to A$. The algebra A is to be interpreted as functions on which space ? What is the interpretation of the ideal $\ker(f)$?
(3) Recall what the *localization of a ring on a multiplicative subset* is. Recall that this is an exact functor. Recall the important example of the localization at a prime ideal of a ring.
(4) Let A be a finite type k-algebra. Let $a \in A$. Recall the localization A_a (so with respect to the multiplicative subset $\{a^n\}_{n\geq 0}$). The ring A_a is to be interpreted as functions on which space ? Is $k[x, y]_y$ a finite type k-algebra ?
(5) Let R be a ring R' and R'' some R-algebras. Recall what is the *tensor product $R' \otimes_R R''$*. What is the R-algebra law on this tensor product ? Which is the universal property of this object as an R-algebra ?
(6) Recall that for a ring R, an ideal I of R, multiplicative subset S of R and an R-algebra $\varphi : R \to A$,

$$R/I \otimes_R A \cong A/IA \quad S^{-1}R \otimes_R A \cong \varphi(S)^{-1}A.$$

(7) Let A and B be finite type k-algebras. The k-algebra $A \otimes_k B$ is to be interpreted as the functions on which space ? Now fix surjections $k[x_1, \ldots, x_n] \to A$ and $k[x_1, \ldots, x_n] \to B$. The k-algebra $A \otimes_{k[x_1,\ldots,x_n]} B$ is to be interpreted as the functions on which space ?
(8) Let R be a ring and M an R-module. Recall what is $\mathrm{Ann}(M)$ and show that if $I \leq \mathrm{Ann}(M)$ then M is naturally a R/I-module. Deduce for example that I/I^2 is an A/I module.

Where are we headed? We will introduce the theory of *schemes*. From the course on algebraic curves, you learned how to interpret *finite type k-algebras* as functions on closed subsets of $\mathbb{A}^n_k$, and saw that the study of such spaces was ultimately related to the algebras of their functions. With the theory of schemes, we will now interpret *any commutative ring as functions on some space.* For example, $\mathbb{Z}$ or any ring of integers can be interpreted as functions on some space, and also rings in finite characteristic. This unveils an all new range of geometric objects. One of the strength of the theory of schemes is that it is a general framework which captures not only the geometry of curves over $\mathbb{C}$ but also the geometry of objects that are more arithmetic in nature. The dictionary between algebra and geometry in the setting that you know and was recalled in a small amount in the preceding exercise will extend to the general setting of schemes.

Solution key.

(5) The universal property of $R' \otimes_R R''$ is that a R-algebra map out of this to an R-algebra S is the same as a pair of maps of R-algebras $R' \to S$ and $R'' \to S$. It is therefore the *coproduct* of R and R' in the category of R-algebras.

(7) $A \otimes_k B$ is to be interpreted as function on the product of the associated closed subspaces of $\mathbb{A}^n_k$ and $A \otimes_{k[x_1,\dots,x_n]} B$ as functions on their intersection in $\mathbb{A}^n_k$.

□

The following exercises are about *sheaves*. Unless specifically mentioned, a sheaf means a *set*-valued sheaf. For a topological space X, $\mathrm{Op}(X)$ denotes the poset of opens of X.

Exercise 2. *$\mathcal{H}om$ sheaf*

Let X be a topological space. Let $\mathcal{F}$ be a presheaf on X and $\mathcal{G}$ be sheaf on X. For any open $U \subset X$, denote by $\mathcal{F}_U$ the presheaf on U defined by $V \mapsto \mathcal{F}(V)$ for any $V \subset U$. Show that the presheaf $\mathcal{H}om : U \mapsto \mathrm{Hom}(\mathcal{F}_U, \mathcal{G}_U)$, where $\mathrm{Hom}(\mathcal{F}_U, \mathcal{G}_U)$ denotes the set of morphisms of (pre)sheaves on U, is a sheaf.

Solution key. We first expose a proof for sheaves of sets. Let (U_i) be an open cover of X.[1] Let $\varphi_i : \mathcal{F}_{U_i} \to \mathcal{G}_{U_i}$ be a collection of morphisms who agree on intersection. We show that it lifts uniquely to a morphism of presheaves $\mathcal{F} \to \mathcal{G}$.

Let V be any open of X. Consider $s \in \mathcal{F}(V)$. Using that $\mathcal{G}$ is a sheaf, that morphisms agree on intersections, and that φ_i is a morphism of presheaves for all i, we get that $(\varphi_{i,V\cap U_i}(s_{V\cap U_i}))$ lifts uniquely to an element of $\mathcal{G}(V)$ that we denote by $\varphi_V(s)$. We want to show that $(\varphi_V : \mathcal{F}(V) \to \mathcal{G}(V))$ is a morphism of presheaves. To see that, note that if $V' \subset V$ and $s \in \mathcal{F}(V)$,

$$\varphi_{V'}(s_{V'})_{|V'\cap U_i} \overset{\text{def. of } \varphi}{=} \varphi_{i,V'\cap U_i}(s_{V'\cap U_i})$$

$$\overset{\varphi \text{ is a morphism of presheaves}}{=} \varphi_{i,V\cap U_i}(s_{V\cap U_i})_{|V'\cap U_i} \overset{\text{def. of } \varphi}{=} \varphi_V(s)_{|V'\cap U_i}$$

so both $\varphi_{V'}(s_{V'})$ and $\varphi_V(s)_{V'}$ restrict on $V' \cap U_i$ to the same element. As $\mathcal{G}$ is a sheaf, the desired equality

$$\varphi_{V'}(s_{V'}) = \varphi_V(s)_{V'} \quad (\varphi \text{ is a morphism of presheaves})$$

follows. Note that for any $V \subset U_i$ we see by definition that $\varphi_V = \varphi_{i,V}$. This shows the existence of the lift.

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[1] This case will suffice; for a general open V we can apply the reasoning to $X = V$ and $\mathcal{F} = \mathcal{F}_{|V}$ and $\mathcal{G} = \mathcal{G}_{|V}$.

As for the unicity note that value on $s \in \mathcal{F}(V)$ of a lift φ' necessarily restricts to $(\varphi_{i,V\cap U_i}(s_{V\cap U_i}))$. Therefore the uniqueness follows from the uniqueness in the sheaf property of $\mathcal{G}$.[2]

We answer now a question asked during TA sessions : can we do this with sheaves with value in an arbitrary category $\mathcal{C}$? The answer is yes and we will do some preliminary definitions. Note that in the above proof there is essentialy three steps: one commutative diagram to show the existence, one to show that this defines a natural transformation, and one argument for the unicity. The proof below is the same pattern.

Let $\mathcal{C}$ be a complete category. A sheaf $\mathcal{F}$ on X with values in $\mathcal{C}$ is a presheaf such that for any open U of X and open covering (U_i) of U, the following[3]

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] & \prod_i \mathcal{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} \mathcal{F}(U_{ij}) \end{tikzcd}$$

is an equalizer diagram. We denote by $\mathrm{Sh}_{\mathcal{C}}(X)$ the full subcategory of $\mathrm{Psh}_{\mathcal{C}}(X) = \mathrm{Fun}(\mathrm{Ouv}(X)^{op}, \mathcal{C})$ consisting of sheaves with values in $\mathcal{C}$.

Now we define the Set-valued presheaf

$$U \mapsto \mathrm{Hom}_{\mathrm{Sh}_{\mathcal{C}}(U)}(\mathcal{F}_U, \mathcal{G}_U)$$

Now we want to show that this pre-sheaf is a sheaf, if we make the hypothesis that $\mathcal{G}$ is a sheaf. To show this, take $(U_i)_{i\in I}$ an open cover of $U \in \mathrm{Ouv}\, X$ and a collection of natural transformations

$$(\alpha^i \colon \mathcal{F}_{U_i} \to \mathcal{G}_{U_i})_{i\in I}$$

such that for all $i, j \in I$ and $W \subset U_{ij}$

$$(\alpha^i_W \colon \mathcal{F}(W) \to \mathcal{G}(W)) = (\alpha^j_W \colon \mathcal{F}(W) \to \mathcal{G}(W)). \tag{1}$$

We need to show that there is a unique natural transformation $\widehat{\alpha} \colon \mathcal{F}_U \to \mathcal{G}_U$ such that restricting this natural transformation to a U_i gives α_i.

Let $V \subset U$ be open. By the universal property of the product, let :

$$\beta_V : \mathcal{F}(V) \to \prod_{i\in I} \mathcal{G}(V \cap U_i)$$

induced by

$$\mathcal{F}(V) \to \mathcal{F}(V \cap U_i) \xrightarrow{\alpha^i_{V\cap U_i}} \mathcal{G}(V \cap U_i).$$

Now we want to consider $\widehat{\alpha}_V \colon \mathcal{F}(V) \to \mathcal{G}(V)$ the unique morphism who would be given the universal property of the following equalizer (because $\mathcal{G}$ is a sheaf) for the cover of V being $(V \cap U_i)_i$. Note that if $V \subset U_i$, by construction, we will have $\widehat{\alpha}_V = \alpha^i_V$.

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[2] If one now wants to show a similar statement for sheaves of abelian groups/rings/etc. one can now argue that to verify that a morphism of presheaves of sets is a morphism of presheaves of abelian groups/rings/etc. it suffices to check it at stalks/locally, which will hold because by construction it will already hold locally.

[3] with the two maps being on component (i, j) once $\prod_k \mathcal{F}(U_k) \to \mathcal{F}(U_i) \to \mathcal{F}(U_{ij})$ and $\prod_k \mathcal{F}(U_k) \to \mathcal{F}(U_j) \to \mathcal{F}(U_{ij})$ the other time

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[d, dashed, "\widehat{\alpha}_V"'] \arrow[dr, "\beta_V"] & & \\
\mathcal{G}(V) \arrow[r] & \prod_{i\in I} \mathcal{G}(V\cap U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} \mathcal{G}(V\cap U_{ij})
\end{tikzcd}$$

To see that this works, we need to show that β_V commutes indeed in this diagram.
This holds, because of the commutative the diagram below, who commutes because $\mathcal{F}$ and $\mathcal{G}$ are functors, that α^i, α^j are natural transformations and that using (1) we have $\alpha^i_{V\cap U_{ij}} = \alpha^j_{V\cap U_{ij}}$.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d] & \mathcal{F}(V\cap U_i) \arrow[d] \arrow[dr, "\alpha^i_{V\cap U_i}"] & & \\
\mathcal{F}(V\cap U_j) \arrow[r] \arrow[dr, "\alpha^j_{V\cap U_j}"'] & \mathcal{F}(V\cap U_{ij}) \arrow[ddrr, "\alpha^i_{V\cap U_{ij}}=\alpha^j_{V\cap U_{ij}}"] & \mathcal{G}(V\cap U_i) \arrow[ddr, bend left] & \\
& \mathcal{G}(V\cap U_j) \arrow[drr, bend right] & & \\
& & & \mathcal{G}(V\cap U_{ij})
\end{tikzcd}$$

So $\widehat{\alpha}_V \colon \mathcal{F}(V) \to \mathcal{G}(V)$ is indeed well defined.
We claim that $(\widehat{\alpha}_V : \mathcal{F}(V) \to \mathcal{G}(V))_{V\subset U}$ is a natural transformation lifting the collection above.
We show that $\widehat{\alpha}$ is natural. This mean we have to show that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[d, "\widehat{\alpha}_{V'}"] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V')
\end{tikzcd}$$

By the universal property of the equalizer (using again that $\mathcal{G}$ is a sheaf), it amounts to prove the commutativity of,

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[r, "\widehat{\alpha}_{V'}"] & \mathcal{G}(V') \arrow[d] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i)
\end{tikzcd}$$

So using the universal property of the product, we need only to verify that for every i :

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[r, "\widehat{\alpha}_{V'}"] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i) \arrow[d] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i) \arrow[r] & \mathcal{G}(V'\cap U_i)
\end{tikzcd}$$

commutes. But this holds because we can insert commutating diagrams inside the diagram above in the following way :

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[ddd, "\widehat{\alpha}_V"'] \arrow[dr] & \mathcal{F}(V') \arrow[r, "\widehat{\alpha}_{V'}"] \arrow[dr] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V' \cap U_i) \arrow[ddd] \\
 & \mathcal{F}(V \cap U_i) \arrow[r] \arrow[dr, "\alpha^i_{V \cap U_i}"'] & \mathcal{F}(V' \cap U_i) \arrow[ddr, "\alpha^i_{V' \cap U_i}"] & \\
 & & \mathcal{G}(V \cap U_i) \arrow[dr] & \\
\mathcal{G}(V) \arrow[r] \arrow[urr] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V' \cap U_i) \arrow[r] & \mathcal{G}(V' \cap U_i)
\end{tikzcd}$$

The intermediate diagrams commute because of the functoriality of $\mathcal{F}$ and $\mathcal{G}$, the naturality of α^i and the definition of $\widehat{\alpha}$.
The unicity of the lift is left to show. Suppose that $\widehat{\alpha'}$ is a lift. Then for any V, and $i \in I$ we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha'}_V"'] & \mathcal{F}(V \cap U_i) \arrow[d, "\alpha^i_{V \cap U_i}"] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V \cap U_i)
\end{tikzcd}$$

Therefore we see that by universal property of $\mathcal{G}(V)$ as an equalizer with respect to the sheaf property and the cover $(U_i \cap V)_i$ of V that $\widehat{\alpha'}_V = \widehat{\alpha}_V$.
□

Exercise 3. *Constant sheaves*
Consider the sheafification of the constant presheaf of $\mathbb{Q}$-vector spaces on the real line $\mathbb{R}$ defined by

$$U \in \mathrm{Op}(\mathbb{R}) \mapsto \mathbb{Q}.$$

We denote by this sheafification $\underline{\mathbb{Q}}$. Compute the value of $\underline{\mathbb{Q}}$ on any open subset of the real line. When the dimension of the $\mathbb{Q}$-vector space $\underline{\mathbb{Q}}(U)$ is finite ? In this case, what is this dimension ?

Solution key. Let S be a set and X a topological space. In what follows we prove that on a connected open subspace U the canonical map $S \to \underline{S}(U)$ is a bijection. We use the following description

$$\underline{S}(U) = \{(s_x) \in \prod_{x \in U} S \mid \forall x \in X \quad \exists U \ni x \quad \forall y, y' \in U \quad s_y = s_{y'}\}$$

and the natural map $S \to \underline{S}(U)$ being the diagonal. Let $(t_x) \in \underline{S}(U)$. Fix $y \in U$ (connected implies non empty). Now note that

$$V_1 = \{x \in U \mid t_x = t_y\} \quad V_2 = \{x \in U \mid t_x \neq t_y\}$$

form a disjoint decomposition of U into open subspaces. As U is connected and $y \in V_1$ we get $V_2 = \emptyset$ and the claim follows.

Now, as any subset U of the real line is a disjoint union of connected open subsets (which is also true for any locally connected space), we get that $\underline{\mathbb{Q}}(U) = \prod_{\pi_0(U)} \mathbb{Q}$ using the sheaf property. This vector space is finite dimensional when U has finitely many connected components and the dimension is then equal to $\pi_0(U)$.

□

Exercise 4. *Sheaves and sections*

(1) Let X and Y be topological spaces, and $f : Y \to X$ a continuous map. Show that the following

$$\mathcal{F}_f(U) = \{s : U \to Y \text{ continuous} \mid f \circ s = \mathrm{id}_U\}$$

defines a sheaf on X. We call it the sheaf of section of f.

(2) Let $\mathcal{F}$ be a sheaf on a topological space X. Define the topological space

$$|\mathcal{F}| = \bigsqcup_{x \in X} \mathcal{F}_x$$

as a set with the finest topology such that for any $U \subset X$ open and $s \in \mathcal{F}(U)$ the section $x \mapsto s_x$ of the canonical map is continuous. Show that $|\mathcal{F}| \to X$ is a local homeomorphism and that the sheaf of section of this map is isomorphic to $\mathcal{F}$.

Solution key. (2) Everything in what follows works for a presheaf, except the isomorphism of sheaves proved at the end – namely this shows that the sheaf of sections is rather isomorphic to the sheafification of the presheaf if we do the construction for a presheaf.

Note first of all that any $s \in \mathcal{F}(V)$ the map $|s|\colon V \to |\mathcal{F}|$ defined by $x \mapsto s_x$ is a section of $p : |\mathcal{F}| \to X$. Note also that

$$|s|(V) = \{s_x \mid x \in V\}$$

is open. Indeed, we need to show by definition of the topology that for any V' open and $t \in \mathcal{F}(V')$

$$|t|^{-1}(|s|(V)) = \{x \in V \cap V' \mid s_x = t_x\}$$

is open. This follows from the following lemma about directed colimits.[4]

Lemma. *Let (A_i) be a directed system of sets and $\varinjlim_i A_i$ the colimit. If $a_i \in A_i$ and $a_j \in A_j$ coincide in the colimit, then there exists k with $i \to k$ and $j \to k$ with the image of a_i and a_j being the same in A_k.*

Proof. One checks that the colimit is given by the quotient of $\bigsqcup_i A_i$ by the relation $(a_i \in A_i) \sim (a_j \in A_j)$ if and only if there exist $i \to k$ and $j \to k$ with a_i and a_k identified in A_k. Once this understood, the lemma follows. □

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[4] As forgetful functors to sets from abelian groups or rings commute with directed colimits, this lemma also applies to directed colimits of abelian groups, rings.

Now, it follows that $p : |\mathcal{F}| \to X$ is continuous. Indeed for an open set U of X we have

$$p^{-1}(U) = \bigcup_{(s,V), s\in\mathcal{F}(V)} |s|(V).$$

Also, we see that for any open V and $s \in \mathcal{F}(V)$ we have $p_{|s|(V)} \circ |s| = \mathrm{id}_V$ and $|s| \circ p_{|s|(V)} = \mathrm{id}_{|s|(V)}$. Therefore p is a local homeomorphism.

We now show that that the sheaf of sections of $|\mathcal{F}| \to X$ is naturally isomorphic to $\mathcal{F}$. Let's write $\mathcal{G}$ for this sheaf of sections.

First, note that a natural map $\mathcal{F} \to \mathcal{G}$ is given by

$$s \mapsto |s|.$$

We want to show that this map is an isomorphism. Note that this is injective because if $|s| = |t|$ it implies that s and t have the same value at every stalk, implying the equality of sections.

Now, for surjectivity: take $x \in X$, any open $U \ni x$ and say $g\colon U \to |\mathcal{F}|$ is a continuous section. Then $g(x) \in \mathcal{F}_x$, so there is an open $V \ni x$ such that $V \subset U$ and there is an $s \in \mathcal{F}(V)$ such that $s_x = g(x)$. Using that g is continuous we get that

$$V' := g^{-1}(|s|(V)) = \{y \in U \mid g(y) = s_y\}$$

is open, and non-empty because $x \in V'$. This shows that

$$g_{V'} = |s_{V'}|.$$

Therefore,

$$\mathcal{F}_x \to \mathcal{G}_x$$

is surjective. Because x is arbitrary, this concludes.

□

Remark. One can promote the construction $\mathcal{F} \mapsto |\mathcal{F}|$ to an equivalence of categories between $\mathrm{Sh}(X)$ and $\acute{\mathrm{E}}\mathrm{t}(X)$ the category of local homeomorphisms over X. For more details, see for example *Manifolds, sheaves, and cohomology* by Wedhorn.

Exercise 5. *Sheafification*
Let X be a topological space and $\mathcal{F}$ a presheaf on X. Show that the natural map $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism at stalks.
Find examples of topological spaces X and presheaves $\mathcal{F}$ on X such that

1. The natural map $\mathcal{F} \to \mathcal{F}^+$ is not injective/resp. not surjective on some non empty open set.
2. An abelian group valued presheaf with $\mathcal{F} \neq 0$ but $\mathcal{F}^+ = 0$.

Solution key. To show that $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism at stalks, we proceed as follows. Note that for any open $U \ni x$ the following projection map

$$\mathcal{F}^+(U) \subset \prod_{x \in U} \mathcal{F}_x \to \mathcal{F}_x$$

will pass to the colimit $(\mathcal{F}^+)_x \to \mathcal{F}_x$. One immediately checks that this is an inverse to the induced map at stalks from $\mathcal{F} \to \mathcal{F}^+$.
For (2) and "not injective" we can take the presheaf on $\mathbb{R}$ with value $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}$ and 0 for any other open.
For "not surjective", take $\mathbb{R}$ and the sheafification of any non-zero abelian group. See "constant" sheaf exercise 3.

□

Exercise 6. *Some sheaves on the circle*
We are using the notation $\mathcal{F}_f$ from exercise 4.

1. Consider the map $e : [0, \frac{3}{2}] \to S^1$ defined by $t \mapsto \exp(2\pi i t)$. Compute all stalks of $\mathcal{F}_e$.
2. Let $\mathcal{O}$ be the presheaf on S^1 defined for $U \in \mathrm{Op}(S^1)$ by

   $$\mathcal{O}(U) = \{U \to \mathbb{R} \text{ continuous}\}$$

   Show that $\mathcal{O}$ is a sheaf. Note that $\mathcal{O}$ is a sheaf of $\mathbb{R}$-algebras by acting pointwise. Show that for every $z \in S^1$, $\mathcal{O}_z$ is a local $\mathbb{R}$-algebra with residue field $\mathbb{R}$.

   Consider now the quotient M of $[0, 1] \times \mathbb{R}$ by identifying $(0, t)$ with $(1, -t)$. Consider the map $\pi\colon M \to S^1$ defined by $\pi([x, t]) = \exp(2\pi i x)$. We also take the notation $\mathcal{F}_\pi = \mathcal{L}$.
3. Show that for every $U \in \mathrm{Op}(S^1)$, $\mathcal{L}(U)$ is an $\mathcal{O}(U)$-module by $\mathcal{O}(U)$ acting on the second component.
4. Show that for every open set $U \subset S^1$ with at least one point missing there is an isomorphism of sheaves $\mathcal{O}_U \cong \mathcal{L}_U$ which respects the module structure on evaluation on each open subset.
5. Show that for every $s \in \mathcal{L}(S^1)$ there exist a $z \in S^1$ such that $s(z) = [z, 0]$.
6. Deduce that there is *no* isomorphism $\mathcal{O} \cong \mathcal{L}$ of sheaves respecting the module structure on each open subset.

Solution key. (1) Note that $e : [0, \frac{3}{2}] \to S^1$ is a local homeomorphism. We claim that the natural evaluation map

$$(\mathcal{F}_e)_z \xrightarrow{\mathrm{ev}_z} e^{-1}(z)$$

is a bijection.[5] Let $x \in e^{-1}(z)$. Let $U \ni Z$ such that $e_{|U}$ is an homeomorphism. Then $e_{|U}^{-1}(z) = x$. This shows surjectivity. If s, t are sections on say $V \ni Z$ and $V' \ni z$ which have the same value on z, say x, then take an open $U \ni x$ such that $e_{|U}$ is an homoemoprhism and $e(U) \subset V \cap V'$. Then $s_{|e(U)}$ and $t_{|e(U)}$ are both the unique inverse to $e_{|U}$. This shows the injectivity.

(2) We show that $\mathcal{O}_z$ is a local $\mathbb{R}$-algebra. We claim that the ideal

$$\{f \in \mathcal{O}_z \mid f(z) = 0\}$$

is the unique maximal ideal. To this end, it suffices to show that the complement consits of the invertible elements. If $f(z) \neq 0$, then there exists a neighbourghood of z where f never vanishes. Therefore $\frac{1}{f}$ is a well defined multiplicative inverse in the stalk.

Some setup and notations for the rest of the exercise.

(a) To avoid confusion, we write the complex number $e(0) = e(1) = 1 \in S^1$ by u.
(b) Denote by $e\colon [0,1] \to S^1$ the quotient map given by $\exp(2\pi i -)$.
(c) The quotient map $p\colon [0,1] \times \mathbb{R} \to M$ gives an homeomorphism

$$p\colon (0,1) \times \mathbb{R} \to \pi^{-1}(S^1 \setminus u).$$

(d) The quotient map $p\colon [0,1] \times \mathbb{R} \to M$ gives an homeomorphism

$$p\colon [0, \frac{1}{2}] \times \mathbb{R} \to \pi^{-1}(S^1_{\geq 0}),$$

where $S^1_{\geq 0}$ denotes the points of the circle with imaginary part positive or zero.

(e) The quotient map $p\colon [0,1] \times \mathbb{R} \to M$ gives an homeomorphism

$$p\colon [\frac{1}{2}, 1] \times \mathbb{R} \to \pi^{-1}(S^1_{\leq 0}),$$

where $S^1_{\leq 0}$ denotes the points of the circle with imaginary part negative or zero.

Let $s \in \mathcal{L}(U)$ be a section. We define a continuous map $\alpha_s\colon e^{-1}(U) \to \mathbb{R}$ such that

$$s(e(t)) = [e(t), \alpha_s(t)].$$

For $t \neq 0, 1$, we define $\alpha_s(t)$ to be the second component of $p^{-1}_{|(0,1)\times\mathbb{R}}(s(e(t)))$, by (c) above. When $t = 0$ and $t = 1$, we extend by continuity: for $t = 0$ we define $\alpha_s(0)$ to be the second component of $p^{-1}_{|[0,\frac{1}{2}]\times\mathbb{R}}(s(u))$ and for $t = 1$ we define $\alpha_s(1)$ to be the second component of $p^{-1}_{|[\frac{1}{2},1]\times\mathbb{R}}(s(u))$.

Note that

$$\alpha_s(0) = -\alpha_s(1)$$

because $s(u) = [0, \alpha_s(0)] = [1, \alpha_s(1)]$.

(3) We define a module structure. We explain how to define the multiplication by scalars, the others operations being defined similarly. Let U be any open of M. Let $f \in \mathcal{O}(U)$ and $s \in \mathcal{L}(U)$. We define

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[5] Note that the following argument holds true for any local homeomorphism $e : X \to Y$.

$f \cdot s$ as follows. We pass to the quotient map $e\colon [0,1] \to S^1$, the following continuous map $[0,1] \to M$

$$t \mapsto [t, f(e(t))\alpha_s(t))].$$

To show that it passes to the quotient we have to show that it agrees on $t = 0$ and $t = 1$. But as

$$f(u)\alpha_s(0) = f(u)(-\alpha_s(1)) = -f(u)\alpha_s(1),$$

this follows from the quotient relation of the Möbius band.

The zero element is the section $s_0 : S^1 \to M$, $s_0(e(t)) = [t, 0]$.

One continue similarly to define the rest of the structure. The key is that the "gluing of the quotient" $(-1)\colon \mathbb{R} \to \mathbb{R}$ is an automorphism of $\mathbb{R}$-modules so that we can "lift" calculations to pointwise calculations in $[0,1] \times \mathbb{R}$. That's why we put the emphasis on that in the above calculation.

(4) For any section $s \in \mathcal{L}(U)$ we have the unique map

$$\mathcal{O}_{|U} \to \mathcal{L}_{|U}$$

that respects the module structure on each open subset of U and sends 1 to s. We claim that if s vanishes nowhere, then this map is an isomorphism. To prove that, we suppose that s vanishes nowhere, and construct an homeomorphism over U

$$\psi_s\colon \pi^{-1}(U) \to U \times \mathbb{R}$$

defined by $[t, \lambda] \mapsto (e(t), \frac{\lambda}{\alpha_s(t)})$. This is well defined by non-vanishing. The inverse is given by $(z, \lambda) \mapsto (\lambda \cdot s)(z)$, where $\cdot$ designates the module structure defined above. Now

$$pr_2\psi_s(-)\colon \mathcal{L}_{|U} \to \mathcal{O}_{|U}$$

gives an inverse to the above map.

We are now left to prove that on any open subset missing a point U, there exist a non-vanishing section. But whenever a point is missing, say $e(t_0) \notin U$ for some $t_0 \in [0,1)$ then we can define the section $U \to M$ by

$$e(t) \mapsto \begin{cases} [t, 1] & t < t_0 \\ [t, -1] & t > t_0 \end{cases}$$

which vanishes nowhere.

(5) Let $s \in \mathcal{L}(S^1)$. By the intermediate value theorem $\alpha_s\colon [0,1] \to \mathbb{R}$ necessarily vanishes at some point because $\alpha_s(0) = -\alpha_s(1)$.

(6) Note that a section $s \in \mathcal{L}(U)$ vanishes at $z = e(t)$ in the sense that $s(z) = [t, 0]$ if and only if $s_z \in \mathfrak{m}_z\mathcal{L}_z$. Note that $1 \in \mathcal{O}(S^1)$ vanishes on no point. By contradiction, the image of 1 by an isomorphism $\mathcal{O} \cong \mathcal{L}$ would not vanish at any stalk, in contradiction with the previous point.

$\square$

Exercise 7. *Skyscraper sheaves*

For any set S and $x \in \mathbb{R}$ show that the following defines a sheaf, for $U \in \mathrm{Op}(\mathbb{R})$

$$x_*S(U) = \begin{cases} S \text{ if } x \in U \\ \mathbb{1} \text{ if } x \notin U, \end{cases}$$

where $\mathbb{1}$ is the set with one element. We call this sheaf the *skyscraper sheaf of S at x*. Compute every stalk of x_*S. Understand and draw the topological space $|x_*S|$ (see exercise 4). Do you understand the name *skyscraper* now ?