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AG II - Schemes and sheaves

Solutions – week 13

Exercise 1. *Using a short exact sequence.*
Let $\iota\colon D \to X$ be an effective Cartier divisor on an integral scheme X. As there is a short exact of sheaves

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \iota_* \mathcal{O}_D \to 0.$$

In consequence there is a long exact sequence in cohomology,

$$(\dots) \to \mathrm{H}^i(X, \mathcal{O}(-D)) \to \mathrm{H}^i(X, \mathcal{O}_X) \to \mathrm{H}^i(D, \mathcal{O}_D) \to (\dots)$$

Let k be a field. Consider $\mathbb{P}^5_k = \mathrm{Proj}(k[x_0, \dots, x_5])$. We consider the closed subscheme

$$X = V_+(x_0^2 + x_1 x_2).$$

You can freely use that X is a Cartier divisor in $\mathbb{P}^5_k$ with ideal sheaf isomorphic to $\mathcal{O}_{\mathbb{P}^5_k}(-2)$.

(1) Show that

$$\mathrm{H}^i(X, \mathcal{O}_X) = 0$$

if $i > 0$ and that $\mathrm{H}^0(X, \mathcal{O}_X) = k$.

(2) Show that for $1 \leq j \leq 3$ we have

$$\mathrm{H}^i\big(X, \mathcal{O}_X(-j)\big) = 0$$

for all $i \geq 0$.

Solution key. We know from the computations in the lecture that

$$\mathrm{H}^0(\mathbb{P}^5_k, \mathcal{O}_{\mathbb{P}^5_k}) = k$$

and $\mathrm{H}^i(\mathbb{P}^5_k, \mathcal{O}_{\mathbb{P}^5_k}) = 0$ for $i > 0$ and that for $1 \leq d \leq 5$ we have

$$\mathrm{H}^i(\mathbb{P}^5_k, \mathcal{O}_{\mathbb{P}^5_k}(-d)) = 0$$

for all $i \geq 0$.

(1) we have a long exact sequence

$$(\dots) \to \mathrm{H}^i(\mathbb{P}^5_k, \mathcal{O}_{\mathbb{P}^5_k}(-2)) \to \mathrm{H}^i(\mathbb{P}^5_k, \mathcal{O}_{\mathbb{P}^5_k}) \to \mathrm{H}^i(X, \mathcal{O}_X) \to (\dots)$$

so the statement follows from the facts recalled above.

(2) We have a short exact sequence tensoring by $\mathcal{O}_{\mathbb{P}^5_k}(-j)$

$$0 \to \mathcal{O}_{\mathbb{P}^5_k}(-2-j) \to \mathcal{O}_{\mathbb{P}^5_k}(-j) \to \iota_* \mathcal{O}_X(-j) \to 0$$

When $1 \leq j \leq 3$, we see that claim also follows from the long exact sequence and the facts recalled above.

□

Exercise 2. *Stability properties of (very-)ample sheaves under tensor product.* Let X be a Noetherian scheme. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on X.

(1) If $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, show that $\mathcal{L} \otimes \mathcal{M}$ is ample.
(2) If $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, deduce that there is a n such that $\mathcal{L}^n \otimes \mathcal{M}$ is ample.
(3) Show that if $\mathcal{L}$ and $\mathcal{M}$ are ample, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

Now suppose that X is an A-scheme where A is a Noetherian ring.

(4) If $\mathcal{L}$ is A-very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is A-very ample.
(5) If $\mathcal{L}$ is ample, then there is a $n_0 > 0$ such that $\mathcal{L}^n$ is A-very-ample for all $n \geq n_0$.

Solution key. (1) First note that if $\mathcal{F}$ and $\mathcal{G}$ are globally generated then $\mathcal{F} \otimes \mathcal{G}$ is also because this is a local claim and the tensor product of two surjective map is surjective. The claim follows.
(2) Follows.
(3) Same.
(4) Choose sections $s_0, \ldots, s_n \in \mathcal{L}(X)$ which defines an A-immersion $X \to \mathbb{P}^n_A$ and $m_0, \ldots, m_m \in \mathcal{M}(X)$ that defines an A-morphism $X \to \mathbb{P}^m_A$. Then the product morphism $X \to \mathbb{P}^n_A \times_A \mathbb{P}^m_A$ is an immersion as immersions are closed under base-change. Now using a Segre embedding concludes: this will be the map induced by tensor sections $s_i \otimes m_j$, and this will be an immersion as a composition of an immersion and a closed immersion.
(5) Follows from previous point and the proposition shown in class that there exists *some* n_0 with $\mathcal{L}^{n_0}$ being A-very ample.

$\square$

Exercise 3. *A Čech cohomology computation.* Let k be a field. Let $U = \mathbb{A}^2_k \setminus 0$. Compute the cohomology of $\mathcal{O}_U$ on U. After showing that $\mathcal{O}_U$ is ample, deduce that Serre vanishing does not hold for U.

Solution key. The Cech complex of $\mathcal{O}_U$ with respect to the open cover $\mathcal{U} = \{D(x), D(y)\}$ is

$$\begin{array}{ccccccc}
C^0(\mathcal{O}_U) & \xrightarrow{\quad d \quad} & C^1(\mathcal{O}_U) & \longrightarrow & \mathcal{C}^2(\mathcal{O}_U) & \longrightarrow & \dots \\
\| & & \| & & \| & & \\
\mathcal{O}_U(D(x)) \times \mathcal{O}_U(D(y)) & & \mathcal{O}_U(D(xy)) & & 0 & & \\
\| & & \| & & & & \\
k[x, x^{-1}, y] \times k[x, y, y^{-1}] & & k[x, x^{-1}, y, y^{-1}] & & & & \\
 & & & & & & \\
(s, t) & \longmapsto & s - t & & & &
\end{array}$$

Thus we have

$$\check{H}^1(\mathcal{U}, \mathcal{O}_U) = \frac{k[x, x^{-1}, y, y^{-1}]}{\mathrm{Im}(d)} = \frac{k[x, x^{-1}, y, y^{-1}]}{k[x, x^{-1}, y] + k[x, y, y^{-1}]} = \bigoplus_{i,j<0} \langle x^i y^j \rangle.$$

Note that $\mathcal{O}_U$ is k-very-ample. Indeed viewing U in $\mathbb{P}^2_k$, the pullback of $\mathcal{O}(1)$ on U is trivial. It now follows that $\mathcal{O}_U$ is k-very ample. Indeed we may extend any coherent sheaf on U on $\mathbb{P}^2_k$ where the tensor by a power of $\mathcal{O}(1)$ is globally generated. Therefore the restriction is also globally generated when restricted to U.

□

Exercise 4. *Curves in $\mathbb{P}^2_k$.* Let k be a field. Let $C = V_+(F)$ for $F \in \mathcal{O}_{\mathbb{P}^2_k}(d)(\mathbb{P}^2_k)$ for a $d \geq 1$.

(1) Show that $\mathrm{H}^0(C, \mathcal{O}_C) \cong k$.
(2) Deduce that any C_1 and C_2 of the above form intersect.
(3) Deduce that $\mathrm{H}^1(C, \mathcal{O}_C)$ is a k-vector space of dimension $\frac{(d-1)(d-2)}{2}$ using the long-exact sequence from exercise 1.

Solution key. For the first item, use that $\mathrm{H}^1(\mathbb{P}^2_k, \mathcal{O}(-d)) = 0$. For the second item, suppose by contradiction that two curves $V_+(F)$ and $V_+(G)$ do not intersect. Then the union is not connected. But the union is described as $V_+(FG)$. Being disconnected, they would be non trivial idempotents in the global sections, a contradiction. We consider

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0,$$

this gives an associated long exact sequence in cohomology. From the lecture we have

$$\dim_k(\mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d))) = \dim_k(\mathrm{H}^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d))) = 0$$

and

$$\dim_k(\mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d))) = \frac{(d-1)(d-2)}{2}.$$

Also we have $\mathrm{H}^i(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) = 0$ for $i > 0$ and global sections being k. All in all we get

$$\dim_k(\mathrm{H}^0(C, \mathcal{O}_C)) = 1 \quad \dim_k(\mathrm{H}^0(C, \mathcal{O}_C)) = \frac{(d-1)(d-2)}{2}.$$

□

Remark. We say that $\frac{(d-1)(d-2)}{2}$ is the *arithmetic genus* of C. Curves of degree 3 have arithmetic genus 1. Smooth ones are called *elliptic curves.* Any smooth curve C over an algebraically closed field k with $\mathrm{H}^1(E, \mathcal{O}_E) = 1$ can be realized as a smooth cubic in $\mathbb{P}^2_k$, see for example Hartshorne III,4.6.

Exercise 5. *Blow-ups, revisited.* Let X be a scheme and $\mathcal{I}$ a quasi-coherent ideal sheaf. We define the *blow-up* of X at $\mathcal{I}$ to be

$$\pi\colon \mathrm{Bl}_{\mathcal{I}}(X) = \underline{\mathrm{Proj}}(\bigoplus_{n\geq 0} \mathcal{I}^n) \to X$$

where $\underline{\mathrm{Proj}}$ denotes the relative Proj of the graded $\mathcal{O}_X$-algebra $\bigoplus_{n\geq 0} \mathcal{I}^n$. Note that for every open affine $\mathrm{Spec}(A) \subset X$, where $\mathcal{I}$ corresponds to I, the pullback of the above is the blow-up $\mathrm{Proj}(\bigoplus_{n\geq 0} I^n)$. The affine blow-up previously introduce in the exercises.

(1) Show that $\mathcal{O}(1)$ of this relative Proj is the ideal sheaf corresponding to the exceptional divisor (the pullback of $V(\mathcal{I})$ along π). Note then that by construction, the exceptional divisor of a blow-up is always Cartier. We denote $\mathcal{O}(1)$ accordingly by $\mathcal{O}(-E)$ below. We also recall that from the Proj construction we have a canonical surjective map of sheaves

$$\pi^*\mathcal{I} \to \mathcal{O}(1) = \mathcal{O}(-E).$$

(2) *Resolving indeterminacies.* Suppose now that X is an S-scheme. Let $\mathcal{L}$ be a line bundle on X. Let $s_0, \ldots, s_n \in \mathcal{L}(X)$ be global sections. Let

$$U = \bigcup_{i=0}^{n} D(s_i) \to \mathbb{P}^n_S$$

be the induced morphism.

(a) *Base locus.* Show that there is a unique ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I}\mathcal{L}$ is the image of the natural map

$$\mathcal{O}_X^{\oplus(n+1)} \xrightarrow{(s_0,\ldots,s_n)} \mathcal{L}.$$

The associated closed subscheme $V(\mathcal{I})$ is denoted by $V(s_0, \ldots, s_n)$ and is called the *base locus of* $s_0, \ldots, s_n$. Furthermore, realize (because $\mathcal{L}$ is locally free), that the natural map $\mathcal{I} \otimes \mathcal{L} \to \mathcal{I}\mathcal{L}$ is an isomorphism. Show also that

$$X \setminus V(\mathcal{I}) = \bigcup_{i=0}^{n} D(s_i).$$

(b) *Resolving maps.* Show that you can extend the partially defined map $X \dashrightarrow \mathbb{P}^n_S$ (totally defined on U) to a map

$$\begin{array}{ccc} \mathrm{Bl}_{\mathcal{I}}(X) & & \\ {\scriptstyle\pi}\downarrow & \searrow & \\ X & \dashrightarrow & \mathbb{P}^n_S \end{array}$$

Hint: In this case, show that $\mathcal{O}(-E) \otimes \pi^*\mathcal{L}$ has $n+1$ induced global generating sections $\pi^* s_i$ as summarized below

$$\pi^*\mathcal{O}_X^{\oplus(n+1)} \xrightarrow{(\pi^* s_0, \ldots, \pi^* s_n)} \pi^*(\mathcal{I}\mathcal{L}) \cong \pi^*\mathcal{I} \otimes \pi^*\mathcal{L} \to \mathcal{O}(-E) \otimes \pi^*\mathcal{L}.$$

(3) *Application to curves.* Let C be a regular curve over k. Let $C \dashrightarrow \mathbb{P}^n_k$ be a partially defined map on a non-empty open set of C. Show that there is a unique extension of the map to $C \to \mathbb{P}^n_k$. *Hint: show that blowing-up a Cartier divisor is an isomorphism.*

Solution key. (1) Note that $\mathcal{O}(1)$ is by definition the ideal sheaf associated to the graded quasi-coherent sheaf

$$\bigoplus_{n \geq -1} \mathcal{I}^{n+1}.$$

Note that forgetting the degree -1, we have an isomorphism to the graded ideal sheaf $\bigoplus_{n \geq 0} \mathcal{I}^{n+1}$ which is the ideal sheaf of the exceptional divisor. Because $\widetilde{-}$ of a graded map which is an isomorphisms in any large enough degree is an isomorphism, we conclude.

(2) (a) We tensor the surjection by $\mathcal{L}^\vee$ to get

$$\mathcal{L}^{\vee \oplus(n+1)} \xrightarrow{(\mathrm{ev}_{s_0}, \ldots, \mathrm{ev}_{s_n})} \mathcal{O}_X.$$

The image is by definition a quasi-coherent ideal sheaf $\mathcal{I}$ with a surjective-injective morphism of sheaves factorization

$$\begin{array}{ccc} \mathcal{L}^{\vee \oplus(n+1)} & \xrightarrow{\mathrm{ev}_{(s_i)}} & \mathcal{O}_X \\ & \searrow & \uparrow \\ & & \mathcal{I} \end{array}$$

Tensoring again by $\mathcal{L}$, and using that $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{I}\mathcal{L}$ is an isomorphism because $\mathcal{L}$ is a line bundle, we get a surjective-injective morphism of sheaves factorization

$$\begin{array}{ccc} \mathcal{O}_X^{\oplus(n+1)} & \xrightarrow{(s_0, \ldots, s_n)} & \mathcal{L} \\ & \searrow & \uparrow \\ & & \mathcal{I}\mathcal{L} \end{array}$$

To show that $\mathcal{I}$ is unique, we actually invoke another construction. If $\mathcal{N} \subset \mathcal{L}$ is a sub-quasi-coherent sheaf, then on a trivializing affine open cover (U_i) we have that this becomes $I_i \subset \mathcal{O}_{U_i}(U_i)$ up to trivializing. But note that the ideal I_i is independent of the choice of trivialization because a different choice of trivialization will only differ by a multiplication by a unit, which leaves an ideal unchanged. So (I_i) glues to an ideal sheaf $\mathcal{I}$ which by construction has the property that $\mathcal{N} = \mathcal{I}\mathcal{L}$. Apply this to the image of $\mathcal{L}^{\vee \oplus (n+1)} \xrightarrow{(\mathrm{ev}_{s_0}, \ldots, \mathrm{ev}_{s_n})} \mathcal{O}_X$.

(b) Everything is in the Hint. Namely note that on the pullback by U the blow-up map is an isomorphism and $\mathcal{O}(E)_{|b^{-1}(U)} \cong \mathcal{O}_U$ so this shows that the map induced by the sections of the Hint actually extend the map defined on U.

□

Exercise 6. *An example of resolution.* Consider k an algebraically closed field and $X = \mathbb{P}^2_k$. Let $x_0, x_1 \in \mathcal{O}_{\mathbb{P}^2_k}(1)$. Then, it defines a partially defined map

$$U = \mathbb{P}^2_k \setminus V_+(x_0, x_1) \to \mathbb{P}^1_k.$$

(1) Show that on k-rational points, this map is to be understood as $[\lambda_0 \colon \lambda_1 \colon \lambda_2] \mapsto [\lambda_0, \lambda_1]$.

Consider the blow-up of $\mathbb{P}^2_k$ at $\mathcal{I}$ the ideal sheaf defining $V_+(x_0, x_1) = [0\colon 0\colon 1]$ – this is the ideal sheaf of the base locus of x_0, x_1, see Exercise 5.b.a. We denote this blow-up by $\pi \colon B \to \mathbb{P}^2_k$.

(2) Write $\frac{x_0}{x_2} = x$ and $\frac{x_1}{x_2} = y$. Show that the exceptional divisor E of this blow-up is isomorphic to $\mathbb{P}^1_k$. More precisely, by computing the blow-up locally in

$$D_+(x_2) = \mathbb{A}^2_{x,y},$$

understand that the exceptional divisor naturally identifies to the projective space of lines through the origin of $\mathbb{A}^2_{x,y}$.[1] In particular, any k-rational point of the exceptional correspond to such a line.

(3) Using Exercise 5, consider the resolved map

$$B \to \mathbb{P}^1_k.$$

Show that a k-rational point of the exceptional corresponding to a line L through $[0 : 0 : 1]$, say given by $[\mu \cdot \lambda_1 \colon \mu \cdot \lambda_2 \colon 1]$ for $\mu \in k$ and $(\lambda_1, \lambda_2) \in k^2 \setminus 0$ is sent to $[\lambda_1 \colon \lambda_2]$ by this resolved map.[2]

Hint: work locally on $D_+(x_2)$. Then understand explicitly the map on standard blow-up charts. Another method is to show that the intersection of E with the closure of $\pi^{-1}(L \setminus [0\colon 0\colon 1])$ in B is precisely the point corresponding to L. Then, because the projection is constant along L, conclude.

Proof. (3) We are looking at the resolution of the map

$$\mathbb{A}^2 \setminus 0 \to \mathbb{P}^1_k$$

[1] Note also that the $\mathbb{P}^1_k$ where we project is naturally understood as $\mathbb{A}^2_{x,y} \setminus 0 / \mathbb{G}_{m,k}$.

[2] One could summarize as follows: the line in the exceptional is sent to itself by the resolved map. This is explained by the fact that the resolved map from the blow-up of $\mathbb{A}^2_k$ at 0 to $\mathbb{P}^1_k$ (resolving the quotient map $\mathbb{A}^2_k \setminus 0 \to \mathbb{P}^1_k$) is equal to the tautological line bundle of $\mathbb{P}^1_k$.

sending (x, y) to $[x, y]$. Blowing up and resolving gives the gluing of the maps

$$k[\frac{x}{y}] \to k[y, \frac{x}{y}]$$

and

$$k[\frac{y}{x}] \to k[x, \frac{y}{x}].$$

The exceptional is given by (y) and (x) in the charts. Restricting to it is the post-composition by the quotient, which gives the identity maps $k[\frac{x}{y}] \to k[\frac{x}{y}]$ and $k[\frac{y}{x}] \to k[\frac{y}{x}]$ which glues to the identity map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ as claimed.

□