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AG II - Schemes and sheaves

Solutions – week 12

Exercise 1. *Very ample divisors.* Let k be a field. Let S be a $\mathbb{N}$-graded ring finitely generated in degree 1 with $S_0 = k$. Denote by $X = \mathrm{Proj}(S)$. Suppose that X is integral and $\mathcal{O}_X(X) = k$.[1]

(1) Show that $\mathcal{O}_X(1)$ is k-very ample.
(2) If $\dim(X) \geq 1$, show that $\mathbb{Z} \xrightarrow{\mathcal{O}_X(1)} \mathrm{Pic}(X)$ is injective.
Hint: If $\mathcal{O}_X(1)$ is torsion, it would imply that $\mathcal{O}_X$ is k-very ample.
(3) If X is normal, deduce that if $0 \neq s \in \mathcal{O}_X(1)(X)$, then $\mathrm{div}(s) \in \mathrm{Cl}(X)$ has infinite order.

Solution key.
(1) Denote by $s_0, \ldots, s_n$ degree 1 elements that $S_0 = k$ generates S as an algebra. Note that $k[x_0, \ldots, x_n] \to S$ sending $x_i \mapsto s_i$ is a graded surjection and therefore induces a closed immersion $\iota\colon X = \mathrm{Proj}(S) \to \mathbb{P}^n_k$. Note that by construction $\iota^*\mathcal{O}_{\mathbb{P}^n_k}(1) \cong \mathcal{O}_X(1)$. The claim now follows.
(2) If $\mathcal{O}_X(1)$ is torsion, meaning that $\mathcal{O}_X(n) \cong \mathcal{O}_X$, it would mean that $\mathcal{O}_X$ is k-very ample. As we suppose that $\mathcal{O}_X(X) = k$, this would imply that X is a point, a contradiction with the dimension hypothesis.
(3) Follows from the injection $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ (normal) and the previous point.

□

Exercise 2. *Homotopy invariance and an instance of Künneth.* Let X be a Noetherian, integral separated and regular in codimension 1 scheme[2]. Let $\pi\colon Y \to X$ be a scheme morphism such that there exists an affine open cover $(U_i = \mathrm{Spec}(A_i))$ of X with $\mathbb{A}^1_{U_i} \cong \pi^{-1}(U_i)$ over U_i.

(1) Follow the steps below to deduce that $\mathrm{Cl}(Y) \cong \mathrm{Cl}(X)$.
 (a) Let t be the image of $t \in A_i[t]$ in $K(Y)$ by an isomorphism $\mathbb{A}^1_{U_i} \cong \pi^{-1}(U_i)$ from the hypothesis for a fixed i. Show that $K(Y) = K(X)(t)$.
 (b) We separate codimension 1 points of Y in two types. Let $y \in Y$ be a point. We say that y is of *type 1* if $\pi(y)$ is of codimension 1. We say that point $y \in Y$ is of *type 2* if $\pi(y)$ is the generic point of X.[3]

[1] This condition follows from previous assumptions if k is algebraically closed.
[2] These hypothesis are the ones to use the notion of *Weil divisors*.
[3] Only these two cases are possible because π is flat and therefore the codimension can only drop as a consequence of going-up.

(i) When y is a point of type 1, show that if $\pi(y) = \mathfrak{p} \in \operatorname{Spec}(A_i)$ then y identifies with the prime ideal $\mathfrak{p}A_i[t] \in \mathbb{A}^1_{U_i} = \operatorname{Spec}(A_i[t])$ up to the isomorphism $\mathbb{A}^1_{U_i} \cong \pi^{-1}(U_i)$ from the hypothesis. From this analysis, deduce that $\mathcal{O}_{Y,y}$ is a DVR.

(ii) Show that points of type 2 are in one to one correspondence with closed points of $\mathbb{A}^1_{K(X)}$. In this case, deduce also that the local rings $\mathcal{O}_{Y,y}$ are DVR's.

(c) Deduce that Y is also regular in codimension 1. Show also that Y is Noetherian, integral separated.

(d) Let y be a point of type 2. Show that y is linearly equivalent to a linear combination of points of type 1. More precisely show that if $(f) \in K(X)[t]$ correspond to y, then show that $y - \operatorname{div}(f)$ is of type 1.

(e) Let $Z \subset X$ be a prime divisor. Show that $\pi^{-1}(Z)$ the topological preimage of Z is a prime divisor with a generic point of type 1. Moreover show that if D is a Weil divisor of $X = \sum n_i Z_i$ then $\pi^{-1}(D) := \sum n_i \pi^{-1}(Z_i)$ cannot be linearly equivalent to a principal divisor of Y unless Z is already principal in X.

(f) Using the above show that the map

$$\pi^* \colon \operatorname{Cl}(X) \to \operatorname{Cl}(Y)$$

sending $\sum n_i Z_i \mapsto \sum_i n_i \pi^{-1}(Z_i)$ is a well defined isomorphism.

(2) Consider $X \times \mathbb{P}^n$ for $n \geq 1$. Using the exact sequence from Week 10, Exercise 3 with the open $X \times D_+(X_0)$, and point (1) of this exercise deduce that we have a split exact sequence (find the splitting)

$$1 \longrightarrow \mathbb{Z} \longrightarrow \operatorname{Cl}(X \times \mathbb{P}^n) \longrightarrow \operatorname{Cl}(X) \longrightarrow 1$$

so that $\operatorname{Cl}(X \times \mathbb{P}^n) \cong \operatorname{Cl}(X) \times \mathbb{Z}$.

Solution key. (1) (a) We may first argue that Y is also integral. It is locally integral so we may only show that it is irreducible. But note that the intersection $\pi^{-1}(U_{ij})$ is never empty being isomorphic to $\mathbb{A}^1_{U_{ij}}$. Therefore one can compute the ring of meromorphic functions of Y on any open, the statement follows.

(b) (i) Note that $\mathfrak{p} = y \cap A_i$ so that $\mathfrak{q} := \mathfrak{p}A_i[t] \subset y$ and they have the image under π. Because the morphism is flat we have

$$\operatorname{ht}(\mathfrak{p}A_i[t]) = \operatorname{ht}(\mathfrak{p}) + \dim(A_i[t]_{\mathfrak{q}}/\mathfrak{p}A_i[t]_{\mathfrak{q}})$$

But $A_i[t]_{\mathfrak{q}}/\mathfrak{p}A_i[t]_{\mathfrak{q}} = k(\mathfrak{p})(t)$. Therefore we see that y and $\mathfrak{q}$ have the same height, so they are equal. Now we want to argue that $\mathcal{O}_{Y,y}$ is a DVR. This follows because $\mathcal{O}_{Y,y}$ is isomorphic to the localization of $A_{i\mathfrak{p}}[t]$ at the (f) where $f \in A_{i\mathfrak{p}}$ is a generator of $\mathfrak{p}$. This is therefore an integral domain which maximal ideal is generated by f, concluding.

(ii) A point of type 2 is precisely a point of codimension 1 in the fiber of the generic point. But this fiber is isomorphic to $A^1_{K(X)}$. This is a 1-dimensional scheme, so the claim follows. The local ring in this case are isomoprhic to localizations at maximal ideals of $K(X)[t]$, which are DVR's.

(c) The above analysis explains that Y is regular in codimension 1. Integral was already touched upon. Noetherian is clear. For separation, it follows from separation of X, and the lemma at the end of the document.

(d) By definition, at every point of type 2, the valuation of $y-\mathrm{div}(f)$ is zero.

(e) The first claim follows from the analysis in (1,b,i) above. Suppose now that

$$D' = \pi^{-1}(D) = \sum n_i \pi^{-1}(Z_i)$$

is principal in Y. So there is some $f \in K(Y)$ such that the above is equal to the divisor associated to f. Up to the using an isomorphism as in (1.a), one can write $f = g(t)/h(t)$ where $g(t), h(t) \in K[t]$. We can suppose that $g(t)$ and $h(t)$ have no common factors. We claim that both are actually of degree zero. Indeed, as there is no type 2 point in D', the valuation of $g(t)$ and $h(t)$ at every maximal ideal of $K[t]$ is trivial. This implies that they are units in $K[t]$, concluding.

(f) The map is well defined because it sends principal divisor to principal divisors. Namely the principal divisor $\mathrm{div}_X(f)$ for $f \in K(X)$ is sent to the principal $\mathrm{div}_Y(f)$ where f is seen in $K(Y)$. This maps is surjective by (d) and injective by (e).

(2) One can argue that $X \times \mathbb{P}^n$ is Noetherian (clear), integral (as above), regular in codimension 1 (locally this is $X \times \mathbb{A}^n$ so as above) and separated using the lemma at the end of the document, or by showing that a product of separated schemes is separated. Consider the pullback

$$\begin{array}{ccc} \mathbb{P}^n_{K(X)} & \longrightarrow & X \times \mathbb{P}^n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K(X)) & \longrightarrow & X \end{array}$$

We claim that we the map $\mathbb{P}^n_{K(X)} \to X \times \mathbb{P}^n$ defines a map on class groups. First, we need to show that any prime divisor is sent to a prime divisor. This is clear, because a prime divisor is a point such that it's only generalization is the generic point, which stays true when pull-backing to $\mathbb{P}^n_{K(X)}$. Also note that a principal divisor will stay principal. Therefore we have a map

$$\mathrm{Cl}(X \times \mathbb{P}^n) \to \mathrm{Cl}(\mathbb{P}^n_{K(X)}).$$

Note that $\mathrm{Cl}(\mathbb{P}^n_{K(X)}) \xrightarrow{\deg} \mathbb{Z}$ (if this was not proved in the lecture, we included a proof at then end of the document.) by the degree map. Now, we are ready to prove the statement. Namely, we use the exact sequence from Exercise 3, Week 10

$$\mathbb{Z} \xrightarrow{V_+(X_0)} X \times \mathbb{P}^n \to \mathrm{Cl}(\mathbb{A}^n_X) \to 0$$

Now, the composition $\mathrm{Cl}(X \times \mathbb{P}^n) \to \mathrm{Cl}(\mathbb{P}^n_{K(X)}) \to \mathbb{Z}$ defines a retraction of the first map. Namely we need to verify that the divisor $V_+(X_0)$ is sent to 1 along the above, which is clear. Using the first part of the exercise, we can also conclude that we have an isomorphism $\mathrm{Cl}(\mathbb{A}^n_X) \to \mathrm{Cl}(X)$, which concludes. □

Exercise 3. *Projective Cone.*
Let S be a $\mathbb{N}$-graded ring finitely generated in degree 1 over S_0. Consider the $\mathbb{N}$-graded ring $S[t]$ with elements of S keeping their grading and with t placed in degree 1. We call $\mathrm{Proj}(S[t])$ with this grading the *projective cone.*

(1) Show that there are natural identifications $V_+(t) = \mathrm{Proj}(S)$ and $D_+(t) = \mathrm{Spec}(S)$. Show furthermore that $V_+(S_+)$ (taken in $\mathrm{Proj}(S[t])$) identifies to $V(S_+)$ in $\mathrm{Spec}(S)$. We denote this closed subscheme by v.
(2) Let $s_0, \ldots, s_n$ be generators of S in degree 1. Show that $\mathrm{Proj}(S[t]) \setminus v$ is covered by the open sets $D_+(s_i)$ and that each open set is isomorphic to $\mathrm{Spec}(S_{(s_i)}[t])$. Deduce that we have a natural map
$$p\colon \mathrm{Proj}(S[t]) \setminus v \to \mathrm{Proj}(S).$$
(3) Let k be an algebraically closed field, and suppose $S_0 = k$. Suppose that S is integral, Noetherian and normal. Suppose that $X = \mathrm{Proj}(S)$ is of dimension ≥ 1. By Exercise 2, note that p^* induces an isomorphism on class groups. Deduce that, if $C = \mathrm{Spec}(S)$ denotes the cone of X then we have an exact sequence
$$1 \to \mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(C) \to 1$$
where the first morphism sends 1 to the class of $\mathcal{O}_X(1)$, and the second is the composition of p^* and the restriction to C. Use the exact sequence from Week 10, Exercise 3 to this effect.

Solution key. (1) The $V_+(t)$ assertion is immediate. For the $D_+(t)$ it amounts to realizing that degree zero elements of S_t are just S. For the last assertion, note that $\mathrm{Proj}(S_0[t])$ identifies with $\mathrm{Spec}(S_0)$.
(2) Note that $(s_0, \ldots, s_n)$ generates S_+ as an ideal. Therefore the claim on the cover follows. Note that the degree zero part of $S[t]_{s_i}$ identifies to $S_{(s_i)}[\frac{t}{s_i}]$ which gives the claim.
(3) We denote by $\overline{X}$ the projective cone.
Note that by Exercise 3, Week 10 we have an exact sequence (where $1 \in \mathbb{Z}$ is sent to $\mathcal{O}_X(1)$ seen as $V_+(t)$, which is prime because X is integral)
$$1 \to \mathbb{Z} \to \mathrm{Cl}(\overline{X}) \to \mathrm{Cl}(C) \to 1$$
because as argued above $D_+(t) = \mathrm{Spec}(S) = C$.

Note that because $v \in \overline{X}$ is at least of codimension 2, we have $\mathrm{Cl}(\overline{X}) \cong \mathrm{Cl}(\overline{X} \setminus v)$. We can now use that p^* induces an isomorphism to conclude.

□

Exercise 4. *Computations of class groups on quadric hypersurfaces.* Suppose that k is algebraically closed and $char(k) \neq 2$. Let $2 \leq r \leq n$. Consider the ring (equipped with the standard grading)

$$S_r = k[x_0, \dots, x_n]/(x_0^2 + \cdots + x_r^2).$$

You can assume that this ring is normal.

(1) Show that up to a linear change of variable, we can suppose that

$$S_r = k[x_0, \dots, x_n]/(x_0x_1 + x_2^2 + \cdots + x_r^2).$$

Denote by $C_r = \mathrm{Spec}(S_r)$ and $X_r = \mathrm{Proj}(S_r)$.

(2) Show that $\mathrm{Cl}(C_r)$ is cyclic when $r \neq 3$
Hint: Consider the prime[4] *divisor* $V(\sqrt{(x_1)})$ *and an exact sequence of class groups.*

(3) Show that $\mathrm{Cl}(C_2) \cong \mathbb{Z}/2\mathbb{Z}$.
Hint: Consider the same exact sequence. Show that $Y = V(\sqrt{(x_1)}) = V(x_1, x_2)$ *is not principal and that* $2Y = 0$.

(4) Show that $\mathrm{Cl}(C_3) \cong \mathbb{Z}$.
Hint: show that after a suitable change of variable we see that $X_r \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$. *Then use Exercise 2.(2) and Exercise 3.(3)*

(5) Show that $\mathrm{Cl}(C_r) \cong 0$ of $r \geq 4$. In particular, S_r is factorial.
Hint: show that (x_1) *is prime in this case and conclude.*

(6) Use the exact sequence of the last point of the above exercise to compute $\mathrm{Cl}(X_r)$ for all $r \geq 2$.

Solution key. (1) Note that $x_0^2 + x_1^2 = (x_0 + ix_1)(x_0 - ix_1)$ where $i \in k$ is a root of -1 in k. Because $char(k) \neq 2$ we can set $y_0 = x_0 + ix_1$ and $y_1 = x_0 - ix_1$ two different variables.

(2) If $r \geq 4$ note that $V(x_1)$ is irreducible because a sum of at least two distinct homogeneous monomials of degree is never a square. Note also that when inverting x_1 evaluating x_0 at $1/x_1 \sum_{i=2}^r x_i^2$ we see that the ring become isomorphic to $k[x_1, x_1^{-1}, x_2, \dots, x_r]$ which is an UFD. Now, using the exact sequence of Week 10, Exercise 3, concludes.

We investigate the case $r = 2$. Here (x_1) is not reduced, but it's reduction (x_1, x_2) is. So one see that the exact same argument apply, using this time the prime divisor (x_1, x_2).

(3) We need to show that (x_1, x_2) is not a principal divisor. This would mean that the ideal (x_1, x_2) is principal. Then it would also be true modulo x_0. But here, we see that (x_1, x_2) is a prime ideal of codimension 2, so it can not be principal. We also want to show that

[4] That's where $r = 3$ is used, so that this ideal is prime.

$2(x_1, x_2) = (x_1)$ has a Weil divisor. In order to see this, we localize at the prime ideal $\mathfrak{p} := (x_1, x_2)$.

$$\left(k[x_0, x_1, x_2]/(x_0x_1 + x_2^2)\right)_{\mathfrak{p}}.$$

Note that $x_0 \notin \mathfrak{p}$ so to compute the above localization we can first localize at x_0, then evaluate x_1 at $\frac{x_2^2}{x_0}$ to see that we are looking at the localization of

$$k[x_0, x_0^{-1}, x_2]$$

at the prime $(\frac{x_2^2}{x_0}, x_2) = (x_2)$ which is principal. Therefore as $x_1 = \frac{x_2^2}{x_0}$ and that x_2 is a generator of the local ring at $\mathfrak{p}$, we see that $V(x_1) = 2\mathfrak{p}$, concluding.

(4) Up to a change of variable we recognize the Segre embedding of $\mathbb{P}_k^1 \times_k \mathbb{P}_k^1$ in $\mathbb{P}_k^3$. We may work with

$$S_r = k[x_0, x_1, x_2, x_3]/(x_0x_1 - x_2x_3)$$

Note that in the exact sequence from the previous exercise, $1 \in \mathbb{Z}$ is sent to the Cartier corresponding to $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$ when we view this in the product. In other words, this is the class of $(1, 1) \in \mathrm{Cl}(\mathbb{P}_k^1 \times_k \mathbb{P}_k^1) = \mathbb{Z} \oplus \mathbb{Z}$. The result follows.

(5) $V(x_1)$ is principal and prime.

(6) We use the exact sequence from the last exercise. For $r \geq 4$, we get that $\mathrm{Cl}(X_r) = \mathbb{Z}$. For $r = 3$, see above. For $r = 2$, we have an exact sequence, were the generator on the left is sent to $V_+(x_1)$

$$1 \to \mathbb{Z} \to \mathrm{Cl}(X_2) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Note that $V_+(x_1) = nV_+(\sqrt{(x_1)})$ for some $n \geq 2$ because $V_+(x_1)$ is irreducible and not reduced. So $V_+(x_1) \in n\,\mathrm{Cl}(X_2)$. Note that the only two possibilities for $\mathbb{Z} \to \mathrm{Cl}(X_2)$ is to be (up to a sign)

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \quad \text{or} \quad \mathbb{Z} \xrightarrow{(1,0)} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

But note that $(1, 0) \notin n(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$ for any $n \geq 2$. This excludes this case. Therefore we can conclude that $\mathrm{Cl}(X_2) \cong \mathbb{Z}$.

□

A criterion for separated schemes.

Lemma. Let X be a scheme. Then X is separated if there exist an affine open (U_i) cover of X such that

(1) $U_{ij} := U_i \cap U_i$ is affine.

(2) The map

$$\mathcal{O}_X(U_i) \otimes \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_{ij})$$

is surjective for every i, j.

Proof. Being separated means that the diagonal morphism $\Delta\colon X \to X \times X$ is a closed immersion. But to check that a morphism is a closed immersion is local on target. So we may check that

$$U_{ij} \to U_i \times U_j$$

is a closed immersion. But because both are affine, the second claim works.

□

Let k be a field. Then the class group on $\mathbb{P}^n_k$ is easily understood. Let D be a prime divisor. We know that it can be represented as $V_+(f)$ for a prime homogeneous ideal f, using that $k[x_0, \dots, x_n]$ is an UFD. Therefore D as a degree, the degree of f, that we denote $\deg(D)$. We define the induced map $\deg\colon \operatorname{Div}(\mathbb{P}^n_k) \to \mathbb{Z}$.

Lemma. The map deg is surjective and the kernel is exactly principal divisors.

Proof. Surjectivity is clear. Suppose that $\operatorname{div}(f)$ is a principal divisor for $f \in k(x_i/x_j)$. Then $f = g/h$ for g and h homogeneous polynomial of the same degree. Therefore we see that principal divisors are in the kernel. The other way around, say that a divisor D is sent to zero. We may write $D = D_0 - D_\infty$ where both are effective. Write f_0 for and f_∞ for polynomials such that $V_+(f_0) = D_0$ and $V_+(f_\infty) = D_\infty$. Then they have the same degree, so $f/g \in K(\mathbb{P}^n_k)$, concluding because $\operatorname{div}(f/g) = D$. □