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EPFL, fall semester 2025
AG II - Schemes and sheaves

Solutions – week 11

Exercise 1. *Effective Cartier divisors.* Let X be an integral scheme. A Cartier divisor on X represented by (f_i, U_i) is said to be *effective* if $f_i \in \mathcal{O}(U_i)$ for every i.

(1) Show, by looking at the ideal sheaf generated by the f_i's, that effective Cartier divisors are in one-to-one correspondence with ideal sheaves $\mathcal{I}$ that are a locally free sheaves of rank 1. We take this point of view in what follows.

(2) Let $\mathcal{L}$ be a locally free sheaf of rank 1. Show that $s\colon \mathcal{O}_X \to \mathcal{L}$ is non-zero if and only if the evaluation $\mathrm{ev}_s : \mathcal{L}^\vee \to \mathcal{O}_X$, defined by $\mathcal{L}^\vee(U) = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{L}_U, \mathcal{O}_u) \ni \varphi \mapsto \varphi(s)$ is injective.

(3) Fix a locally free sheaf $\mathcal{L}$ of rank 1. Deduce the following bijection,

$$\frac{(\Gamma(X,\mathcal{L}) \setminus \{0\})}{\mathcal{O}_X(X)^\times} \to \{\text{Effective Cartier divisors } \mathcal{I} \text{ on } X \text{ with } \mathcal{I} \cong \mathcal{L}^\vee\}$$

that sends the class of a section s to $\mathrm{Im}(\mathrm{ev}_s)$.

(4) Suppose that $\mathcal{O}_X(X)$ is a field. Show that if $\mathcal{L}$ is a locally free sheaf of rank 1 such that $\mathcal{L}$ and $\mathcal{L}^\vee$ have a non zero section, then $\mathcal{L} \cong \mathcal{O}_X$. *Hint: in this case both $\mathcal{L}$ and $\mathcal{L}^\vee$ correspond to effective Cartier divisors.*

(5) Additionally assume that X is normal, Noetherian and integral. *Two Weil divisors are called linearly equivalent if their difference is the divisor of some rational function.* Let D be a Weil divisor on X. Show that map sending $f \in \Gamma(X, \mathcal{O}_X(D))$ to $\mathrm{div}(f) + D$ gives a one to one correspondence

$$\frac{\Gamma(X,\mathcal{O}_X(D)) \setminus \{0\}}{\mathcal{O}_X(X)^\times} \to \{\text{Effective Weil divisors linearly equivalent to } D\}.$$

Careful, hypothesis does not imply that $\mathcal{O}_X(D)$ defined as

$$\mathcal{O}_X(D)(U) = \{g \in K(X) \mid g \neq 0, (div(g) + D) \cap U \text{ is effective}\}$$

is a line bundle. So you have to prove it independently of item (3).

Remark. Let X be a projective k-scheme where k is algebraically closed. We will show later in the course that $\Gamma(X, \mathcal{O}_X) = k$ as in the case of the projective space. Using (3) of Exercise 1, we see, for $\mathcal{L}$ a line bundle on X, that

$$\mathbb{P}(\Gamma(X,\mathcal{L}))(k) = \{\text{Effective Cartier divisors } \mathcal{I} \text{ on } X \text{ with } \mathcal{I} \cong \mathcal{L}^\vee\}$$

Therefore $\mathbb{P}(\Gamma(X,\mathcal{L}))$ gives a natural k-scheme structure on the set on the right hand side. This set is called *the complete linear system of $\mathcal{L}$*.

Solution key. (1) If (f_i, U_i) is an effective Cartier divisor, then we see that defining $f_i\mathcal{O}_{U_i} \subset \mathcal{O}_{U_i}$ defines a sub-ideal sheaf of $\mathcal{O}_X$ by gluing because f_i/f_j are units in functions on the intersection by assumption so $f_i\mathcal{O}_{U_{ij}} = f_j\mathcal{O}_{U_{ij}}$. Reciprocally given a locally free ideal sheaf $\mathcal{I}$, we know that there is an open cover (U_i) with $\mathcal{I}_{U_i} = f_i\mathcal{O}_{U_i}$. Now, (f_i, U_i) defines an effective Cartier divisor. Say we choose other generators (on a possible different open cover, but we deal with this case by taking a common refinement) $\mathcal{I}_{U_i} = f_i'\mathcal{O}_{U_i}$. Then there is $g_i \in \mathcal{O}_{U_i}(U_i)^\times$ with $f_i = g_i f_i'$. This implies that $(f_i, U_i) = (f_i', U_i)$ has a Cartier divisor.

(2) This is a local check, so without loss of generality $\mathcal{L} = \mathcal{O}$ and $X = \mathrm{Spec}(A)$ is affine with A integral. So we are saying that $a \in A$ is non-zero if and only if

$$\begin{array}{c} A \xrightarrow{1\mapsto \mathrm{id}} \mathrm{Hom}_A(A, A) \xrightarrow{\mathrm{ev}_a} A \\ \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\cdot a} \end{array}$$

the multiplication by a is injective.

(3) We produce an inverse. Say $\psi\colon \mathcal{I} \cong \mathcal{L}^\vee$. Then apply the $\mathrm{Hom}_{\mathcal{O}}(-, \mathcal{O})$ functor gives a map $s_\psi\colon \mathcal{O} \to \mathcal{I}^\vee \to \mathcal{L}$. Because an automorphism of a line bundle is always given by an element[1] in $\mathcal{O}_X(X)^\times$, it is clear that this inverse map does not depend on the choice of the isomorphism ψ.

(4) If both $\mathcal{L}$ and $\mathcal{L}^\vee$ have non-zero global sections, then say that $\mathcal{L} = \mathcal{I}$ an invertible ideal sheaf without loss of generality. But then $\mathcal{I}$ has a non-zero global section. Because we supposed that $\mathcal{O}_X(X)$ is a field, we see that 1 generates this ideal, concluding.

(5) About the inverse map, if D_1 is effective and linearly equivalent to D, this means that there exists $f \in K(X)$ such that $\mathrm{div}(f) + D = D_1$. So by definition f is a global section of $\mathcal{O}(D)$. Further details ommited. □

Exercise 2. *Extension principle for normal schemes.* Let k be a field. Let $X \to \mathrm{Spec}(k)$ be a normal and integral k-scheme of finite type and let $Y \to \mathrm{Spec}(k)$ a proper k-scheme. Let $U \subset X$ be an open subscheme. Say we have

$$f\colon U \to Y$$

a morphism of k-schemes. Show that we can extend f to a map $g\colon V \to Y$ where V is an open of X containing all generic points of codimension 1 irreducible closed subschemes(=Weil Divisors).

Remark. This generalize the *extension principle* from Exercise 1, Week 7. The proof is very similar!

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[1] An automorphism $\mathcal{L} \to \mathcal{L}$ is given locally by elements in $\mathcal{O}_{U_i}(U_i)^\times$ where $\mathcal{L}$ is trivial, but these will automatically glue. Indeed on the intersection they will be both equal to the restriction of the morphism.

Solution key. Take $x \in X$ a codimension 1 point. Because X is normal, we have that $\mathcal{O}_{X,x}$ is a DVR. So we can use the valuation criterion for properness to have a map $f_x\colon \operatorname{Spec}(\mathcal{O}_{X,x}) \to Y$ as seen below.

$$\begin{array}{ccc} \operatorname{Spec}(K(X)) & \longrightarrow & Y \\ \downarrow & \overset{f_x}{\nearrow} & \downarrow \\ \operatorname{Spec}(\mathcal{O}_{X,x}) & \longrightarrow & \operatorname{Spec}(k) \end{array}$$

Now say $W \ni f_x(x)$ is an open affine of Y. Because open are stable by generalization, note that $f_x^{-1}(W) = \operatorname{Spec}(\mathcal{O}_{X,x})$. So we have a factorization $f_x\colon \operatorname{Spec}(\mathcal{O}_{X,x}) \to W$. The ring $\mathcal{O}(W)$ is a finite type k-algebra because Y is proper over k. Say $a_1, \dots, a_n$ are generators. Then the image of a_i in $\mathcal{O}_{X,x}$ is of the form $\frac{f_i}{g_i}$ with $f_i, g_i \in \mathcal{O}(U_x)$ where U_x is an affine open neighborhood of x. We also have $g_i \notin x$, when we see x as a prime in $\mathcal{O}(U_x)$. Therefore up to localizing $\mathcal{O}(U_x)$ at the product of g_i's we can suppose that $g_i^{-1} \in \mathcal{O}(U_x)$. In the end, we see that there is an open neighborhood U_x of x in X and a map $f_x\colon U_x \to Y$ which agrees with f generically. Do this procedure for every $x \in X$ of codimension 1. Let

$$V = U \cup \bigcup_{x \in X} U_x$$

Now, glue the maps $(f, f_x)_{x \in X}$, which is possible because they all agree generically by construction. As the target is separated, this implies that the map agrees on intersection, by Exercise 5, week 7. □

Exercise 3. *Divisors on regular curves.* Let k be an algebraically closed field. We say that C is a *regular k-curve over k* is a one dimensional separated, integral and regular scheme over k. Weil (=Cartier in this case) divisors are then of the form

$$D = \sum_i n_i x_i$$

for x_i being closed points of C. We define the *degree* of a divisor $D = \sum_i n_i x_i$ to be

$$\deg(D) = \sum_i n_i \in \mathbb{Z}.$$

Let $f\colon C' \to C$ a finite k-morphism between regular k-curves. We define the *pullback* of an irreducible divisor (=closed point)

$$f^* x = \sum_{y \in C'_{cl} \text{ s.t. } x = f(y)} v_y(f^\sharp(t_x))y.$$

where $f^\sharp$ denotes the induced map at the local ring. Here, t_x denotes a generator of $\mathfrak{m}_x$ – this well defined because the choice of a generator is up to a unit. We extend f^* by linearity to $\operatorname{Div}(C)$.

(1) Show that the pullback of a principal divisor is principal, implying that f^* factors through

$$f^*\colon \operatorname{Cl}(C) \to \operatorname{Cl}(C').$$

(2) Show that if the degree of the map ($= [K(C')\colon K(C)]$) is d, then $\deg(f^*D) = d\deg(D)$. *Hint: it suffices to show the claim for $D = x$ a closed point by linearity.*

(3) Assume now that C is also proper. Using the extension principle from Exercise 1, Week 7 (or exercise 2 above) show that for every $t \in K(C)\setminus k$ we have a map $f_t\colon C \to \mathbb{P}^1_k$ from the inclusion $k(t) \subset K(C)$ such that $f_t^*(0-\infty) = (t)$ where 0 denotes $V(t)$ in $\mathrm{Spec}(k[t]) \subset \mathbb{P}^1_k$ and ∞ denotes $V(1/t)$ in $\mathrm{Spec}(k[1/t]) \subset \mathbb{P}^1_k$. deduce that $\deg((t)) = 0$, and that therefore deg factor through

$$\deg\colon \mathrm{Cl}(C) \to \mathbb{Z}.$$

Solution key. (1) One sees that the pullback of $\mathrm{div}(g)$ for some $g \in K(C)$ is given by $\mathrm{div}(f^\sharp(g))$ where here $f^\sharp$ denotes the induced map at field of fractions.

(2) Note that a dominant map $f\colon C' \to C$ between regular curves is necessarily and flat. Indeed for $c \in C'$ closed, and looking at stalks we have

$$\mathcal{O}_{C,f(c)} \to \mathcal{O}_{C',c}$$

an injection, so the right hand side is a torsion-free module on the left hand side. But the left hand side is a PID, and torsion-free modules over a PID are flat.

Therefore, the fiber at x is a finite k-algebra of dimension d. Indeed the map $C' \to C$ is finite flat of degree d. So the algebra of the fiber is finite dimensional k-algebra of degree d. Because k is algebraically closed, such algebras are isomorphic to

$$\prod k[t]/(t_i)^n_i$$

which has much factors has the set theoretic cardinality of the fiber and necessarily $\sum n_i = d$.

(3) Assume that the map given by f is finite and flat (which is true see the remark).

Note that by construction we have $f_t^* \operatorname{div}_{\mathbb{P}^1}(t) = \operatorname{div}_C(t)$ by the first point of the exercise. Now, on $\mathbb{P}^1$, the function t has unique zero at zero and an unique pole at ∞. So $\operatorname{div}_{\mathbb{P}^1}(t) = 0 - \infty$. Therefore

$$\deg(\operatorname{div}_C(t)) = \deg(f_t^* \operatorname{div}_{\mathbb{P}^1}(t)) = \deg(f_t)\underbrace{\deg(0-\infty)}_{=0} = 0$$

□

Remark. Point (2), and therefore the argument in (3), relied on the finite and flatness of the map. This is true, see for example Exercise 3, week 14. In the solutions of this exercise, we also show another proof of (3) independent of the construction of this morphism. Also, one can show that for any dominant map $C' \to C$ between integral regular and proper curves over k, then one can realize the map as follows: take $K(C) \to K(C')$ the induced map. For any open affine $\mathrm{Spec}(A) \subset C$ consider the integral closure of A in $K(C')$, denoted A'. The one shows that the map $C' \to C$ is the gluing of the maps $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ this map is affine and finite by the properties of the normalization for finite type domains over a field. This is a way to show that any map between integral regular and proper curves over k is finite.

Exercise 4. *Closed subschemes of* Proj. Let R be a graded ring finitely generated over R_0 in degree 1. Let $P = R_0[x_0, \dots, x_n]$ be the graded polynomnial ring where x_i's are placed in degree 1.

(1) Show, using the sheaf property on the cover $(D_+(x_i))_{i=0}^n$ that the natural map
$$P_m \to \Gamma(\mathbb{P}^n_{R_0}, \mathcal{O}_{\mathbb{P}^n_{R_0}}(m))$$
is an isomorphism for any $m \in \mathbb{Z}$.

(2) Show that there is a closed immersion
$$\mathrm{Proj}(R) \to \mathbb{P}^n_{R_0}$$
for some $n \in \mathbb{N}$, which comes from a chosen graded surjection $R[x_0, \dots, x_n] \to R$. We fix a such for the rest of the exercise.

(3) Let $Z \to \mathbb{P}^n_{R_0}$ a closed subscheme given by a quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^n_{R_0}}$. Define
$$I := \bigoplus_{m \in \mathbb{N}} \Gamma(\mathbb{P}^n_{R_0}, \mathcal{I}(m)).$$
which is an ideal of P using that $\mathcal{I}(m) \subset \mathcal{O}_{\mathbb{P}^n_{R_0}}(m)$ and (1). Show, using that the natural map $\widetilde{I} \to \mathcal{I}$ is an isomorphism (see the lecture) that the natural map
$$\mathrm{Proj}(P/I) \to \mathbb{P}^n_{R_0}$$
is identified with $Z \to \mathbb{P}^n_{R_0}$.

(4) Now say that $Z \to \mathrm{Proj}(R)$ is a closed-subscheme. Combine (2) and (3) to deduce that there is a graded ideal $J \subset R$ such that
$$\mathrm{Proj}(R/J) \to \mathrm{Proj}(R)$$
is identified with $Z \to \mathrm{Proj}(R)$.

Remark. In the end the ideal constructed in the way suggested by the exercise can be described as follows. Recall that any $r \in R_n$ can be seen as a section in $\Gamma(\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)}(n))$[2]. Now the homogeneous elements of degree n of the ideal defined as suggested in the above exercise are exactly the elements that becomes zero when pulling back to Z when seen a global section of $\mathcal{O}_{\mathrm{Proj}(R)}(n)$.

Solution key. (1) Consider the cover of $\mathrm{Proj}(P)$ by $D_+(x_i)$ for $i = 0, \dots, n$. Recall that
$$\mathcal{O}(m)_{|D_+(x_i)}(D_+(x_i)) = x_i^m P_{(x_i)}.$$

First note that the natural map is given by sending $f \in P_m$ to global section defined by $f = x_i^m \frac{f}{x_i^m} \in x_i^m R_{(x_i)}$.

[2]There is always the natural map $R_n \to \Gamma(\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)}(n))$ which might not be an isomorphism. But using this map we can consider elements of R_n as sections of $\mathcal{O}_{\mathrm{Proj}(R)}(n)$.

By the sheaf property, a global section of $\mathcal{O}(m)$ corresponds to a collection $f_i/x_i^{n_i}$ where $\deg(f_i) = m+n_i$ that agrees on intersections. Because we work in a polynomial ring, we can suppose that x_i does not divide f_i. Agreeing on intersections says that

$$f_i x_j^{n_j} = f_j x_i^{n_i}.$$

Because x_i, x_j is a regular sequence we deduce that $n_j = n_i = 0$. Therefore we deduce that $f_i = f_j = f$ an homogeneous element of degree m, concluding.

(2) We use functoriality of Proj for a a fixed surjection

$$R_0[x_0, \dots, x_n] \to R.$$

(3) It suffices to show the claim locally, so we may take the pullback to $D_+(x_i)$. The closed immersion $\mathrm{Proj}(P/I) \to \mathbb{P}^n_{\mathbb{R}_0}$ now corresponds to

$$R_0[\frac{x_j}{x_i}] \to R_0[\frac{x_i}{x_j}]/(I)_{(x_i)}.$$

On the other hand, because the natural map $\widetilde{I} \to \mathcal{I}$ is an isomorphism, we have that $(I)_{(x_i)} = \widetilde{I}(D_+(x_i)) \subset R_0[\frac{x_j}{x_i}]$, which concludes.

(4) We use (2) to get a closed immersion $Z \to \mathrm{Proj}(R) \to \mathbb{P}^n_{R_0}$. We use (3) to get an homogeneous ideal $I \subset P$ such that $\mathrm{Proj}(P/I) \to \mathbb{P}^n_{R_0}$ correspond to Z. Note that elements in the kernel K of the fixed surjection

$$R_0[x_0, \dots, x_n] \to R.$$

are necessarily in the ideal I: this is because any element of I is sent to zero when pullbacked to Z. Therefore we can now define J to be $J := I/K \subset R$ to conclude.

□

Exercise 5. *Segre embedding.* In this exercise we use that sections of line bundles correspond to morphisms to a projective space, see the lecture.
Let k be a ring. Denote the projection of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ to the first and second factor by p_1 and p_2 respectively. View the first and second copy of $\mathbb{P}^1$ in the product as $\mathrm{Proj}(k[x_0, x_1])$ and $\mathrm{Proj}(k[y_0, y_1])$ respectively. Show that the global sections $p_1^*(x_i) \otimes p_2^*(y_j)$ for $0 \leq i, j \leq 1$ of $p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$ give a closed embedding of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ in $\mathbb{P}^3_k$.

Exercise 6. *Fibre dimension (of coherent sheaves).*
Let X be a Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Let

$$\varphi \colon X \to \mathbb{N}$$

be defined as $\varphi(x) = \dim_{k(x)}(\mathcal{F} \otimes_{\mathcal{O}_X} k(x))$.
Nakayama's lemma may be useful for the following.

(1) Show that φ is upper semi-continuous meaning that for any $n \geq 0$

$$\{x \in X \mid \varphi(x) \geq n\}$$

is closed.

(2) If $\mathcal{F}$ is locally free and X connected show that φ is constant.

(3) Show that if X is reduced and connected show that $\mathcal{F}$ is locally free if and only φ is constant.

Solution key. Because each question is local, say $X = \operatorname{Spec}(A)$, where A is Noetherian and we work with M global sections of $\mathcal{F}$, which is a finitely generated A-module. Let $\mathfrak{p} \in \operatorname{Spec}(A)$. For (1), note that if $M(\mathfrak{p})$ is of dimension n, say with basis $m_1, \ldots, m_m$, then we have a surjective map by Nakayama

$$A_{\mathfrak{p}}^n \to M_{\mathfrak{p}}.$$

Find some $a \in A$ such that this surjections lifts to a map

$$A_a^n \to M_a.$$

The coker K of this map is finitely generated and satsfies $K_{\mathfrak{p}} = 0$. Therefore we may localize further to have $K_b = 0$ for some $b \in A$ and concluding that we have a surjective map

$$A_b^n \to M_b.$$

This implies that complements of sets in the statement are open.
For (2), note that φ is continuous to the discrete topology on $\mathbb{N}$ if $\mathcal{F}$ is locally free. Therefore only one fiber can be non-empty because fibers are disjoint opens and the union of all fibers cover the space.
As for (3), proceed as in (1) to get a surjective map

$$A_b^n \to M_b.$$

An element in the kernel is a vector $(a_1, \ldots, a_n)$ where each element is in every prime ideal of A_b. Indeed, for any prime ideal $\mathfrak{p}$ of A_b, looking at

$$k(\mathfrak{p})^n \to M(\mathfrak{p})$$

we have a surjective map between to $k(\mathfrak{p})$-vector spaces of the same dimension so also injective. Because A_b is reduced, the intersection of all primes is the zero ideal, concluding.

□