

### Solutions – week 10

**Exercise 1.** *Line bundles and exact sequences.* Let  $X$  be a scheme and  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \text{Qcoh}(X)$ . Let  $\mathcal{L}$  be locally free sheaf of rank 1, *i.e.* a *line bundle*.

(1) Show that

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

is exact if and only if

$$0 \rightarrow \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{L} \rightarrow \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{L} \rightarrow \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{L} \rightarrow 0$$

is.

Let  $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$  be finite locally free sheaves. Show that if

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$$

is exact if and only if

$$0 \rightarrow \mathcal{E}_3^\vee \rightarrow \mathcal{E}_2^\vee \rightarrow \mathcal{E}_1^\vee \rightarrow 0$$

is.

Let  $\mathcal{E}$  be a finite locally free sheaf of constant rank  $n$ . We define

$$\det(\mathcal{E}) := \bigwedge^n \mathcal{E}.$$

(3) Show that  $\det(\mathcal{E})$  is a line bundle. Namely show that  $\det(R^{\oplus n}) \cong R$  where  $R$  is a ring.

(4) Let  $\varphi: R^{\oplus n} \rightarrow R^{\oplus n}$  be an  $R$ -module map, where  $R$  is a ring. Show that  $\varphi$  induces a morphism

$$\det(\varphi): \det(R^{\oplus n}) \rightarrow \det(R^{\oplus n})$$

which is given by the multiplication by the determinant of the matrix defining  $\varphi$ .

(5) Let  $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$  be finite locally free sheaves of constant rank. Suppose that

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$$

is an exact sequence. Then, show that there is an induced isomorphism

$$\det(\mathcal{E}_2) \cong \det(\mathcal{E}_1) \otimes \det(\mathcal{E}_3).$$

**Exercise 2.** *Divisors that are not Cartier.* Let  $k$  be a field and  $X = V(xy - zw)$  in  $\mathbb{A}_k^4$ . Note that  $X$  is integral and regular in codimension 1.

(1) Show that the closed subsets in  $X$  defined by  $x = z = 0$  and  $x = w = 0$  are prime divisors that are not Cartier. Denote by  $D_z$  and  $D_w$  these divisors.

(2) Show that  $D_z + D_w$  is a Cartier divisor.

*Solution key.* Note that  $D_z$  and  $D_w$  are prime Weil divisors isomorphic to  $\mathbb{A}_k^2$  because

$$k[x, y, z, w]/(xy - zw, x, z) = k[y, w] \quad k[x, y, z, w]/(xy - zw, x, w) = k[y, z].$$

If  $I_z = (x, z)$  or  $I_w = (x, w)$  where to define Cartier divisors, then these ideals would be locally principal, in the sense that in sufficiently small affine open sets, these ideals would be principal in the ring of functions of these opens. In particular, in each local ring these ideals would be principal. But at the local ring at the origin  $\mathfrak{m}$ , note that  $\mathfrak{m}/\mathfrak{m}^2$  is a  $k$ -vector space of dimension 4, and the basis is given by the images of  $x, y, z, w$ . If  $I_z$  or  $I_w$  where to be generated by one element in this local ring, then the  $k$ -vector space spanned by the images of  $x, z$  and  $x, y$  respectively in  $\mathfrak{m}/\mathfrak{m}^2$  would be of  $k$ -dimension 1, a contradiction.

Now, we show that  $V(x) = D_z + D_w$ . We want to compute the valuation of  $x$  at every local ring of a codimension point. Note that topologically  $V(x) = V(x, z) \cup V(x, w)$ . Indeed

$$k[x, y, z, w]/(xy - zw, x) = k[x, y, z, w]/(x, zw).$$

So the only codimension primes where  $x$  will have some non-trivial valuation are  $D_z$  and  $D_w$ . But one checks that  $x$  is a generator of both the local rings of  $D_z$  and  $D_w$ , concluding.  $\square$

**Exercise 3.** *Exact sequence for class groups.* Let  $X$  be an integral separated scheme which is regular in codimension 1. Let  $Z$  be a proper closed subset of  $X$  and  $U = X \setminus Z$ .

- (1) Show that  $\text{Cl}(X) \rightarrow \text{Cl}(U)$  defined by  $\sum n_i D_i \mapsto \sum n_i (D_i \cap U)$  is surjective.
- (2) If  $\text{codim}(Z, X) \leq 2$ , show that that this map is also injective.
- (3) If  $\text{codim}(Z, X) = 1$  and  $Z$  is irreducible, show that there is an exact sequence

$$\mathbb{Z} \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 1$$

where  $\mathbb{Z} \rightarrow \text{Cl}(X)$  send 1 to  $Z$ .

- (4) Let  $k$  be a field. Let  $Z$  be the zero set of an irreducible homogeneous polynomial of degree  $d$  in  $\mathbb{P}_k^n$ . Deduce that  $\text{Cl}(\mathbb{P}_k^n \setminus Z) \cong \mathbb{Z}/d\mathbb{Z}$ .

*Solution key.* The last statement follows from example seeing that that  $V_+(F)$  for  $F$  irreducible homogeneous of degree  $d$  correspond to  $\mathcal{O}(d)$  via the identification of Picard groups and Cartier class groups, and the  $\mathcal{O}(d)$  is  $d$  times the generator of the Picard group which is infinite cyclic.  $\square$

**Exercise 4.** *Extension of coherent sheaves.* The goal is to show that if  $X$  is a Noetherian scheme,  $U$  an open subset and  $\mathcal{F}$  is a coherent sheaf on  $U$ , then there is a coherent sheaf  $\mathcal{G}$  on  $X$  such that  $\mathcal{G}|_U \cong \mathcal{F}$ .

- (1) Show that on a Noetherian scheme  $X$  and  $\mathcal{F}$  coherent sheaf, then if

$$\sum_i \mathcal{F}_i = \mathcal{F}$$

where  $(\mathcal{F}_i)_{i \in I}$  are sub-coherent sheaves, then there exist a finite refinement  $J \subset I$  such that  $\sum_{j \in J} \mathcal{F}_j = \mathcal{F}$ .

- (2) Show that on a Noetherian affine scheme, every quasi-coherent sheaf is the direct colimit of its coherent sub-sheaves. *Hint: Use the equivalence of categories with modules on global sections.*
- (3) Let  $X$  be affine and  $\iota: U \rightarrow X$  be an open subscheme. Show the claim in this case. *Hint: Show that  $\iota_*\mathcal{F}$  is quasi-coherent, and then use a combination of (1) and (2) to conclude.*
- (4) Show the claim in the general case of the statement of the exercise by induction on the number of open affines that are required to cover  $X$ . (Being covered by one open affine being the base case of the induction, and is the previous point. The rest is an induction play, see Hint.) *Hint: Say  $X = X_1 \cup X_2$  where  $X_1$  and  $X_2$  are open subschemes that can be covered by strictly less open affines than  $X$ . By induction extend  $\mathcal{F}_{X_1 \cap U}$  to a coherent sheaf  $\mathcal{G}_1$  defined on  $X_1$ . By gluing  $\mathcal{F}$  and  $\mathcal{G}_1$  it defines a coherent sheaf  $\mathcal{G}'$  defined on  $X_1 \cup U$ . Now, extend  $\mathcal{G}'|_{X_2 \cap (X_1 \cup U)}$  to a coherent sheaf  $\mathcal{G}_2$  on  $X_2$ . Conclude by gluing  $\mathcal{G}_1$  and  $\mathcal{G}_2$  to a coherent sheaf on  $X$ .*

As an application, show that any quasi-coherent sheaf on a Noetherian scheme is a direct colimit of sub-coherent sheaves.

*Solution key.* (1) Let  $\mathcal{F}$  be a coherent sheaf on a Noetherian scheme  $X$ . Suppose that

$$\sum_i \mathcal{F}_i = \mathcal{F}$$

where  $(\mathcal{F}_i)_{i \in I}$  are sub-coherent sheaves. Then there exist a finite refinement  $J \subset I$  such that  $\sum_{j \in J} \mathcal{F}_j = \mathcal{F}$ . Indeed, as we can cover  $X$  by *finitely* many open affines, we may prove that we can find a finite refinement for each open affine. But now this follows just from the fact that a coherent sheaf on a Noetherian affine scheme amounts to a finite module  $M$ . Each generator of  $M$  is a finite sum of sections of the  $\mathcal{F}_i$ 's.

- (2) Recall that a Noetherian affine scheme  $\text{Spec}(A)$ , quasi-coherent sheaves are equivalent to  $A$ -modules and coherent sheaves are equivalent to finite  $A$ -modules. Therefore to check that

$$\bigcup \mathcal{F}_\alpha = \mathcal{F}$$

it suffices to check it on global sections. But it amounts to say that an  $A$ -module is the union of its sub finite  $A$ -modules.

- (3) Cover  $U$  by *finitely* many open affine schemes  $(U_i)_i$ . Note that because an affine scheme is separated, intersections  $U_{ij}$  are also affine. We may use that affine morphism preserves quasi-coherence, and that quasi-coherent sheaves are stable by kernels. Denote by

$\iota_i: U_i \rightarrow X$  and  $\iota_{ij}: U_{ij} \rightarrow X$  inclusions. Now remark that

$$\iota_*\mathcal{F} = \ker \left( \bigoplus_i \iota_{i*}\mathcal{F} \rightarrow \bigoplus_{ij} \iota_{ij,*}\mathcal{F} \right)$$

where the map sends  $(f_i) \mapsto (f_i - f_j)$ .

We therefore know that this sheaf is the union of its coherent subsheaves by (a)

$$\bigcup \mathcal{F}'_\alpha = \iota_*\mathcal{F}.$$

Because  $\mathcal{F}_U$  is coherent, by the compactness remark, there exists finitely many  $\alpha_1, \dots, \alpha_n$  such that  $\mathcal{F}_U$  is the sum of the  $\mathcal{F}'_{\alpha_i, U}$ . If we set  $\mathcal{F}'$  to be the sum of these we get the claim.

- (4)  $\mathcal{G}$  is the union of its coherent sub-modules, say  $\mathcal{G}_\alpha$ . By the compactness remark above, we may sum finitely many such that  $\sum_i \mathcal{G}_{\alpha_i, U} = \mathcal{F}$ .
- (d) We proceed by induction on the number of affine schemes that can cover  $X$ . If  $X$  is affine, we are done. Otherwise, we can write

$$X = X_1 \cup X_2$$

with  $X_1$  and  $X_2$  open subschemes that can be written as union of strictly less open affines. Find a coherent sheaf on  $X_1$  that we denote by  $\mathcal{F}'_1 \subset \mathcal{G}_{X_1}$  that extends  $\mathcal{F}|_{X_1 \cap U}$  from  $X_1 \cap U$  to  $X_1$  by induction. Note that this defines a coherent sheaf on  $X_1 \cup U$  by gluing  $\mathcal{F}'_1$  and  $\mathcal{F}$ . Denote this sheaf by  $\mathcal{F}'_1$ . Now extend  $\mathcal{F}'_1|_{X_2 \cap (U \cup X_1)}$  from  $X_2 \cap (U \cup X_1)$  to  $X_2$  by induction to a coherent sheaf  $\mathcal{F}'_2$ . Now remark that  $\mathcal{F}'_2$  and  $\mathcal{F}'_1$  glue to the desired sheaf  $\mathcal{F}'$ .

- (5) Note the proof above can be conducted to prove the following statement. Suppose that  $\mathcal{H}$  is a quasi-coherent sheaf on  $X$  such that  $\mathcal{F} \subset \mathcal{H}_U$ . Then there exists a sub-coherent sheaf  $\mathcal{G} \subset \mathcal{H}$  such that  $\mathcal{G}|_U = \mathcal{F}$ . From this slightly stronger statement, the application follows.

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