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AG II - Schemes and sheaves

Hand-in 5

Exercise to hand in. *Trivialization of tangent bundle for cubics.* (Due Wednesday December 3, 12:00) Please write your solution in TEX.

(1) Let k be a field. Recall that $\Omega_{\mathbb{P}^n_k/k}$ is a locally free sheaf of rank n. Its restriction to $D_+(x_i)$ is free with basis elements $\{d(x_j/x_i)\}_{j\neq i}$:

$$\Omega_{\mathbb{P}^n_k/k}(D_+(x_i)) = \bigoplus_{i\neq j} k[x_j/x_i]_{i\neq j} d(x_j/x_i).$$

On $D_+(x_ix_j)$, where $i \neq j$, compute the base change matrix relating these two bases.

(2) Using the bases above and the base change matrix, prove that $\Omega_{\mathbb{P}^n_k/k}$ does not have any nonzero global sections.

(3) In the hand in exercise of week 6, we proved that the tangent sheeaf of a special cubic curve was trivial. In this exercise, we prove the same result for any cubic curve, but in a different way.

Let $C = \mathrm{Proj}(k[x,y,z]/f)$, where f is a irreducible homogeneous polynomial of degree three, and suppose that C is smooth over k. In other words, C is a smooth cubic curve in $\mathbb{P}^2_k$. Prove that $\Omega_C := \Omega_{C/k}$ is isomorphic to $\mathcal{O}_C$ following the steps below:

(a) Show that the ideal sheaf of $\mathbb{P}^2_k$ defining C is isomorphic to $\mathcal{O}_{\mathbb{P}^2_k}(-3)$.

(b) Let $\iota : C \to \mathbb{P}^2_k$ be the closed immersion. Show that the conormal sequence looks like

$$\iota^*\mathcal{O}_{\mathbb{P}^2_k}(-3) \to \iota^*\Omega_{\mathbb{P}^2_k} \to \Omega_C \to 0.$$

Prove that the left most arrow in the conormal sequence is in fact injective.

(c) Using determinants (Ex 1, week 10) and the Euler sequence (Ex 5, Week 9), deduce that $\Omega_C \cong \mathcal{O}_C$.

Note that using the same method one can compute Ω_X for any smooth hypersurface X in $\mathbb{P}^2_k$.

Solution key. We explain the proof of (3).

(a) We have an isomorphism of graded modules sending 1 to f

$$k[x,y,z](-3) \to fk[x,y,z].$$

Note also that we have a short exact sequence of graded modules

$$0 \to fk[x,y,z] \to k[x,y,z] \to k[x,y,z]/f \to 0.$$

Taking the exact functor $\widetilde{-}$ on both we deduce that the ideal sheaf $\mathcal{I}$ defining C is isomorphic to $\mathcal{O}_{\mathbb{P}^2_k}(-3)$.

(1) Locally, we have an exact sequence with projective cokernel by the Jacobian criterion, which gives the claim using that $\iota^*\mathcal{O}_{\mathbb{P}^2_k} = \mathcal{I}/\mathcal{I}^2$.
(2) Taking determinant of an appropriate twist of the Euler sequence, we get that
$$\omega_{\mathbb{P}^2_k} \cong \mathcal{O}_{\mathbb{P}^2_k}(-3).$$
Using that determinants commutes with pullback and the sequence above we get that
$$\Omega_C \otimes \iota^*\mathcal{O}_{\mathbb{P}^2_k}(-3) \cong \iota^*\mathcal{O}_{\mathbb{P}^2_k}(-3)$$
which concludes.

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