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AG II - Schemes and sheaves

Solutions – week 14

Exercise 1. *Conics.* Let k be a field. Let $C = V_+(F) \subset \mathbb{P}^2_k$ for F a degree 2 homogeneous polynomial. Suppose that C is *geometrically integral*, meaning that $C_{\overline{k}}$ is integral.

(1) Let $\nu\colon C' \to C_{\overline{k}}$ the normalization of $C_{\overline{k}}$. Consider the exact sequence

$$0 \to \mathcal{O}_{C_{\overline{k}}} \to \nu_*\mathcal{O}_{C'} \to K \to 0$$

where K is defined to be the cokernel. Using the long exact sequence in cohomology, show that $K = 0$, and deduce that $C_{\overline{k}}$ is regular.[1]

(2) Show that $C(k) \neq \emptyset$ if and only if $C \cong \mathbb{P}^1_k$.

(3) Consider the closed subscheme of $\mathbb{P}^2_{\mathbb{Z}}$

$$C := V_+(X_0^2 + 5X_1^2 + 7X_2^2) = \operatorname{Proj}\left(\frac{\mathbb{Z}[X_0, X_1, X_2]}{(X_0^2 + 5X_1^2 + 7X_2^2)}\right) \subset \mathbb{P}^2_{\mathbb{Z}}.$$

For which prime numbers p do we have $C_p \cong \mathbb{P}^1_{\mathbb{F}_p}$?

Solution key. Note that from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2_k}(-2) \to \mathcal{O}_{\mathbb{P}^2_k} \to \iota_*\mathcal{O}_C \to 0$$

we get that $\mathrm{H}^0(C, \mathcal{O}_C) = k$ and $\mathrm{H}^1(C, \mathcal{O}_C) = 0$. The same calculation holds for $C_{\overline{k}}$.

(1) Note that K is supported where ν is not an isomorphism, therefore in dimension zero. Therefore $K = 0$ if and only $\mathrm{H}^0(C_{\overline{k}}, K) = 0$. But as $\mathrm{H}^1(C_{\overline{k}}, \mathcal{O}_{C_{\overline{k}}}) = 0$, we have an exact sequence

$$0 \to \mathrm{H}^0(C_{\overline{k}}, \mathcal{O}_{C_{\overline{k}}}) \to \mathrm{H}^0(C', \mathcal{O}_{C'}) \to \mathrm{H}^0(C_{\overline{k}}, K) \to 0$$

where the presence of the group in the middle is justified because ν is affine (finite). But both $C_{\overline{k}}$ and C' are proper, and thus projective curves over $\overline{k}$, integral, so the first two groups are domains that are finite dimensional algebras over $\overline{k}$, therefore necessarily equal to $\overline{k}$. Furthermore, the first arrow is a $\overline{k}$-algebra morphism so it has to be the identity map. We deduce that $K = 0$ and then that $\mathcal{O}_{C_{\overline{k}}} \to \nu_*\mathcal{O}_{C'}$ is an isomorphism. Because ν is affine, this implies that ν is an isomorphism and that $C_{\overline{k}}$ is normal, therefore regular because it is of dimension 1.

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[1] This implies that $C_{\overline{k}}$ is smooth over $\overline{k}$, implying that C is smooth over k. In particular, C is regular.

(2) If $C(k) \neq \emptyset$ then let $x \in C(k)$. Consider $\mathcal{O}_C(x)$ the line bundle associated. We have an exact sequence

$$0 \to \mathcal{O}_C(-x) \to \mathcal{O}_C \to \iota_* k \to 0$$

where $\iota x \to C$ the inclusion. Note that the right hand side is k because x is supposed to be k-rational. We deduce, tensoring by $\mathcal{O}_C(x)$, that there is an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(x) \to \iota_* k \to 0.$$

As $\mathrm{H}^1(C, \mathcal{O}_C) = 0$, we get that $\mathrm{H}^0(C, \mathcal{O}_C(x))$ is of dimension 2 as a k-vector space. So we have an induced map $U \to \mathbb{P}^1_k$ given by two sections. Because C is regular integral and proper over k, we can extend the map to a map $g\colon C \to \mathbb{P}^1_k$. From Exercise 3.(6) of this sheet we know that this map is finite and flat. Because $g^*\mathcal{O}_{\mathbb{P}^1_k}(1) \cong \mathcal{O}_C(x)$ and that $\deg(g^*\mathcal{O}_{\mathbb{P}^1_k}(1)) = \deg(g)\deg(\mathcal{O}_{\mathbb{P}^1_k}(1)) = \deg(g) = \deg(\mathcal{O}_C(x)) = 1$ because x is k-rational point, we get that g is finite flat of degree 1, *i.e.* an isomorphism.

(3) We first show that for any prime number not equal to 5 and 7, the curve C_p has rational points. Namely on $D_+(X_0)$ we are looking at the equation

$$1 + 5x^2 = 7y^2.$$

For $x \in \mathbb{F}_p$ note that the left hand side ranges over $\frac{p+1}{2}$ values. Same goes for the right hand side with $y \in \mathbb{F}_p$. If by contradiction this equation has no solutions, then there would $p+1$ elements in $\mathbb{F}_p$, a contradiction. Now we study the equation prime by prime.

(a) *The behavior for $p = 2$.* The equation modulo 2 is

$$X_0^2 + X_1^2 + X_2^2 = (X_0 + X_1 + X_2)^2.$$

We see that affine locally on $D_+(X_0)$ we have the equation $(1 + \frac{X_1}{X_0} + \frac{X_2}{X_0})^2$ which show that the (geometric) fiber is non-reduced.

(b) *The behavior for $p = 5$.* The equation modulo 5 is

$$X_0^2 - 3X_2^2.$$

By the remark above, because 3 is not a square in $\mathbb{F}_5$, this polynomial is irreducible, and therefore C_5 is integral, in particular, reduced. As 3 acquires a square-root α over $\mathbb{F}_{25}$, the equations becomes

$$(X_0 - \alpha X_2)(X_0 + \alpha X_2).$$

Therefore the base change to $\mathbb{F}_{25}$ is not integral. Namely it is the union of two reduced irreducible components, which are copies of $\mathbb{P}^1_{\mathbb{F}_{25}}$ intersecting at $(X_0, X_2) = [0 : 1 : 0]$. The same reasoning holds for the further base change to the algebraic closure. However it is reduced and geometrically reduced.

(c) *The behavior for $p = 7$.* The equation modulo 7 is

$$X_0^2 - 2X_1^2.$$

But note that 3 is a square root of 2 modulo 7. Therefore we have

$$X_0^2 - 2X_1^2 = (X_0 - 3X_1)(X_0 + 3X_1).$$

Therefore C_7 is not integral being the union of two reduced irreducible components, which are copies of $\mathbb{P}^1_{\mathbb{F}_7}$. These two components intersect at $(X_0, X_1) = [0 : 0 : 1]$. Same holds for base change to the algebraic closure. However it is reduced and geometrically reduced.

(d) *The behavior for* $p \neq 2, 5, 7$. Because $-(5X_1^2 + 7X_2^2)$ is never a square in $\overline{\mathbb{F}}_p[X_0, X_1]$,[2] we see that C_p and $C \times \operatorname{Spec}(\overline{\mathbb{F}}_p)$ are irreducible.

We can conclude that $C_p \cong \mathbb{P}^1_{\mathbb{F}_p}$ if and only if $p \neq 2, 5, 7$.

□

Exercise 2. *Complements of ample divisors are affine.* Let k be a field. Let $X \to \operatorname{Spec}(k)$ be a projective k-scheme. Let D be an ample effective Cartier divisor. Show that $X \setminus D$ is affine.

Proof. Without loss of generality, D is k-very-ample, up to replacing D by nD for a big enough $n \in \mathbb{N}$. Now $\mathcal{O}_X(D)$ is a k-very-ample line bundle. Take a basis $s_0, \cdots, s_n$ where $s_0 = 1_D$, meaning the meromorphic function $1 \in \mathcal{O}_X(D)$, which vanishes exactly at D as a section of this line bundle. So $X_{s_0} = X \setminus D$. But now considering the closed immersion (because $\mathcal{O}_X(D)$ is a k-very-ample line bundle)

$$\varphi \colon X \xrightarrow{[s_0 : \cdots : s_n]} \mathbb{P}^n_k$$

we have that $\varphi^{-1} D_+(x_0) = X_{s_0} = X \setminus D$. Because a closed immersion is affine and stable by pullback, we deduce that $X \setminus D$ is closed in $D_+(x_0)$, therefore affine. □

Exercise 3. *Curves.* Let k be an algebraically closed field. We fix C a smooth connected projective curve over k. We let[3]

$$g := h^1(C, \mathcal{O}_C)$$

the *genus* of the curve C.

(1) *Riemann-Roch.* Show that for any $\mathcal{L} \in \operatorname{Pic}(C) \cong \operatorname{Cl}(C)$ we have[4]

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g.$$

Redo the proof if you have seen it in class.

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[2] Indeed if it was, say $g^2 = -(5X_1^2 + 7X_2^2)$ then the g would divide the partial derivative of $5X_1^2 + 7X_2^2$ with respect to X_1 and X_2 by the Leibniz rule. So g would divide X_1 and X_2 implying that g is a unit.

[3] For a coherent sheaf $\mathcal{F} \in \operatorname{Coh}(C)$ we denote $h^i(C, \mathcal{F}) = \dim_k(H^i(C, \mathcal{F}))$.

[4] The degree of a line bundle is the degree of the corresponding class of divisors in the class group.

(2) Let $x \in C(k)$. Show that there exists a rational function $f \in K(C)$ regular everywhere except at x.

(3) Let U be a non-empty open. Show that there is a rational function regular only at points of U.

(4) Suppose that U is a *strict open* of C. Construct a morphism $f\colon C \to \mathbb{P}^1_k$ with $f^{-1}(D_+(x_0)) = U$.

(5) Let $Z := C \setminus U$. Show that there is an effective divisor D with support exactly Z with D being ample. Deduce that any strict open in an integral regular proper curve is affine.

(6) Let $C' \to C$ be a dominant map between smooth connected proper k-curves. Show that the map is affine, and finite[5] flat of degree $[K(C')\colon K(C)]$.

(7) Show that if there is a closed point $P \in C(k)$ with $\mathrm{H}^0(C, \mathcal{O}_C(P)) = 2$, then $C \cong \mathbb{P}^1_k$ over k.

(8) Show that for any effective divisor D, we have $h^0(C, \mathcal{O}(D)) \leq \deg(D)+1$. Show that equality happens if and only if $D = 0$ or $g = 0$.

Solution key. We will use throughout that Serre duality gives

$$h^i(\mathcal{L}^\vee \otimes \omega_C) = h^{1-i}(\mathcal{L})$$

for $i = 0, 1$ and any line bundle $\mathcal{L}$.

Note that if D is effective, then we have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \iota_* \mathcal{O}_D \to 0$$

where $\dim_k(\mathcal{O}_D) = \deg(D)$. This is because we have an exact sequence

$$0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \to \iota_* \mathcal{O}_D \to 0,$$

that we tensor by $\mathcal{O}_C(D)$ – note that any line bundle on D is trivial because D is of dimension zero.[6] Therefore we can ignore any twists by line bundle on the right hand side.

Note also that the exercise is written so that the goal is to show that if $C' \to C$ is a dominant map between smooth connected proper k-curves, then the map is affine, and finite flat of degree $[K(C')\colon K(C)]$. Note that the theory of degree of line bundles and divisors relies on the fact that the degree of principal divisors is zero. This may have been proven using point (7), and if so this may introduce a circular reasoning. We show that we don't need point (7) to show this. Let $f \in K(C)^\times$. Let $(f) = D_0 - D_\infty$ the principal divisor associated to (f). We want to show that $\deg(D_0) = \deg(D_\infty)$. But note that $\mathcal{O}_C \cong \mathcal{O}_D(D_0) \otimes \mathcal{O}_D(-D_\infty)$. Therefore $\mathcal{O}_C(-D_0) \cong \mathcal{O}_C(-D_\infty) =: \mathcal{L}$. Recall that the dimension of $\mathcal{O}_{D_\infty}$ and $\mathcal{O}_{D_0}$ as k-vector spaces is the degree of those divisors. Therefore from the long exact sequence on cohomology coming from

$$0 \to \mathcal{L} \to \mathcal{O}_C \to \iota_{D_0 *} \mathcal{O}_{D_0} \to 0 \quad 0 \to \mathcal{L} \to \mathcal{O}_C \to \iota_{D_\infty *} \mathcal{O}_{D_\infty} \to 0$$

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[5] Once you get that the map is affine, finiteness is still to show, This does not follow immediately from the fact that it is finite at the generic fiber. Properness of the map helps. See the extra exercise at the end of the sheet.

[6] Line bundles only depend on the underlying topological space of the scheme and a finite type k-scheme of dimension zero has a finite an discrete underlying topological space.

and using the additivity of Euler characteristic we get that

$$\deg(D_\infty) = \deg(D_0) = \chi(\mathcal{O}_C) - \chi(\mathcal{L}).$$

(1) We first prove the formula for anti-effective divisors, by induction on the degree. The degree zero amounts to $\mathcal{O}_C$, where it holds by definition of g and because $\mathrm{H}^0(C, \mathcal{O}_C) = k$. Now for $\mathcal{O}_C(-D)$ where D is effective. Let x be a point of the support, so that $\mathcal{O}_C(D - x)$ is of degree strictly less. We have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(x) \to \iota_* k \to 0$$

so tensoring by $\mathcal{O}(-D)$ gets

$$0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C(x - D) \to \iota_* k \to 0.$$

Now using that the long exact sequence in cohomology has zero Euler characteristic (as any exact sequence of vector spaces) we get

$$h^0(\mathcal{O}_C(x - D)) - h^1(\mathcal{O}_C(x - D)) - 1 = h^0(\mathcal{O}_C(-D)) - h^1(\mathcal{O}_C(-D))$$

so induction settles the claim.

Now, if D is any effective divisor, tensoring by $\mathcal{O}(-D')$ the aforementioned exact sequence for any other effective divisor D' gets

$$0 \to \mathcal{O}_C(-D') \to \mathcal{O}_C(D - D') \to \iota_* \mathcal{O}_D \to 0$$

Now using that the long exact sequence in cohomology has zero Euler characteristic we get

$$h^0(\mathcal{O}_C(D-D')) - h^1(\mathcal{O}_C(D-D')) = h^0(\mathcal{O}_C(-D')) - h^1(\mathcal{O}_C(D')) + \deg(D) = \deg(D-D') + 1 - g.$$

(2-3) Let U be an open and $\{x_1, \cdots, x_n\}$ it's complement. We prove a better version of (2). Namely we show that there is a meromorphic function $f \in K(C)$ with pole only at x_i and no-zeros at x_j for $i \neq j$. Using point (5) of the lemma at the end of the document, we see that we can take n big enough so that $h^1(\mathcal{O}_C(nx_i - x_j)) = 0$ and $h^1(\mathcal{O}_C(nx_i)) = 0$. Therefore $h^0(\mathcal{O}_C(nx_i)) = h^0(\mathcal{O}_C(nx_i - x_j)) + 1$. So

$$\mathrm{H}^0(C, \mathcal{O}_C(nx_i - x_j)) \subset \mathrm{H}^0(C, \mathcal{O}_C(nx_i))$$

is a k-vector subspace of codimension 1, consisting of meromorphic functions in $\mathcal{O}_C(nx_i)(C)$ with a zero at x_j when $i \neq j$ and meromoprhic funtions with pole order at x_i being at most $n - 1$ when $i = j$. The union of this finite number of hyperplanes will never cover all the k-vector space $\mathcal{O}_C(nx_i)(C)$. So we may pick an $f_i \in \mathcal{O}_C(nx_i)(C)$ with a pole of order n and not $n - 1$ at x_i and no zeros at x_j when $i \neq j$.

Now we can deduce the statement of (3) taking the product of all f_i's.

(4) Follow the proof above with the same notations and take f as the product of $f_i's$. Then f defines a map of k-schemes $U \to \mathbb{A}^1_k \to \mathbb{P}^1_k$. Then, it uniquely extends to a map $C \to \mathbb{P}^1_k$. Note that the pre-image of ∞ is exactly where f has poles. But f has poles at $x_1, \ldots, x_n$. Therefore we deduce that because that $f^{-1}(\mathbb{A}^1_k) = U$.

(5) With the help of the Lemma at the end of the document point (7), we see that any effective divisor on a curve is ample. This concludes using Exercise 2.

(6) Such a map will always be proper. Indeed, if X is proper over k and Y is separated over k, and $f\colon X \to Y$ is a k-map, factor f with it's graph $X \to X \times_k Y \to Y$. The first map is a closed immersion and the second is proper. So f is proper. Now let $U \subset C$ be an affine open. It has to be strict because proper and affine over k means finite over k, see Extra exercise. But C is of dimension 1. The pre-image of U has to be strict: indeed $C' \to C$ is dominant and closed so surjective, so it can factor through a strict subspace. By the preceding point, the pre-image is affine.

Now, the map is proper and affine, therefore finite by the Extra exercise. Note that this is flat because on stalks we have torsion free modules over PID's, which are flat.

(7) Let $f, g \in \mathrm{H}^0(C, \mathcal{O}_C(P))$ a basis. Then it defines a map $U \to \mathbb{P}^1_k$. This map extend uniquely to a dominant map $g\colon C \to \mathbb{P}^1_k$. Now by (6) g is finite and flat. We want to show that is of degree 1. Because $g^*\mathcal{O}_{\mathbb{P}^1_k}(1) \cong \mathcal{O}_C(P)$ and that $\deg(g^*\mathcal{O}_{\mathbb{P}^1_k}(1)) = \deg(g)\deg(\mathcal{O}_{\mathbb{P}^1_k}(1)) = \deg(g) = \deg(\mathcal{O}_C(P)) = 1$ because P is k-rational point, we get that g is finite flat of degree 1, *i.e.* an isomorphism.

(8) The key here is to use Serre duality – namely we always have $\omega_C(-D) \subset \omega_C$ for any effective divisor D. So $h^0(\omega_C(-D)) \leq h^0(\omega_C)$. Therefore by Serre duality we have $h^0(\omega_C(-D)) = h^1(\mathcal{O}_C(D)) \leq g$. Therefore the first assertion follows from Riemman-Roch. If $D = 0$, the equality is clear. If $g = 0$, this forces $\mathrm{H}^1(C, \mathcal{O}(D)) = 0$ for any effective divisor, this follows from applying the long exact sequence to

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \iota_*\mathcal{O}_C \to 0.$$

We now make a remark, when $g \geq 1$ the canonical line bundle ω_C is globally generated. Indeed, this is equivalent to $\omega_C(-P) \to \omega_C$ being not surjective on H^0 for every point $P \in C(k)$, see point (1) of Lemma at the end of the document. But note that by Riemman-Roch and Serre duality

$$h^0(\mathcal{O}_C(P)) - \underbrace{h^0(\omega_C(-P))}_{=h^1(\mathcal{O}_C(P))} = 2 - \underbrace{g}_{=h^0(\omega_C)}$$

So if $\omega_C(-P) \to \omega_C$ is an isomorphism on H^0, then the above gives

$$h^0(\mathcal{O}_C(P)) = 2$$

which in turn implies that $C \cong \mathbb{P}^1_k$ and $g = 0$.

Now we proceed to prove that if D is effective and non-trivial and $h^0(\mathcal{O}_C(D)) = \deg(D) + 1$, then $g = 0$. We consider the long exact sequence coming from

$$0 \to \omega_C(-D) \to \omega_C \to \iota_*\mathcal{O}_D \to 0.$$

Suppose that $g \geq 1$. Then the map $\omega_C \to \iota_*\mathcal{O}_D$ can not be zero on H^0, otherwise D would be in the base locus of ω_C (see lemma at the end of the document), and we just argued that ω_C is globally generated. But then it forces the map $\omega_C(-D) \to \omega_C$ to be not surjective on H^0, implying that

$$h^0(\omega_C(-D)) < g = h^0(\omega_C).$$

But using Riemman-Roch and Serre duality, we get

$$h^0(\mathcal{O}_C(D)) - (\deg(D) + 1) + g < g$$

rearranging to

$$h^0(D) < \deg(D) + 1.$$

□

Exercise 4. *More on elliptic curves.*
Let E be a smooth connected projective curve over an algebraically closed field k. Suppose that

$$h^1(E, \mathcal{O}_E) = 1.$$

(1) Using Serre duality and Riemann-Roch, show that $\Omega_{E|k} \cong \mathcal{O}_E$.

(2) *Weierstrass equation.* Fix a k-rational point $e \in E(k)$. Such a pair (E, e) is called an *elliptic curve.* Suppose now that the characteristic of the field is not 2 or 3. Show that one can find $x \in H^0(E, \mathcal{O}(2e)) \backslash k$, $y \in H^0(E, \mathcal{O}(3e)) \setminus H^0(E, \mathcal{O}(2e))$ and coefficients $a, b \in k$ with

$$y^2 = x^3 + ax + b.$$

(3) *Weierstrass embedding.* Using that $\mathcal{O}(3e)$ is very ample, deduce that there is a closed embedding in $\mathbb{P}^2_k$

$$E \to V_+(Y^2 Z = X^3 + aXZ^2 + bZ^3) \subset \mathbb{P}^2_k.$$

What is the image of e in coordinates?

Solution key. (1) We have that $\deg(\omega_C) = 2g - 2$ in general using Serre duality and Riemman-Roch. So here we have $\deg(\omega_C) = 0$. But also $h^0(\omega_C) = 1$. Therefore ω_C is effective and of degree zero, which implies that $\omega_C \cong \mathcal{O}_C$.

(2) By the Lemma at the end of the document, point (5), we have that $h^0(E, \mathcal{O}(ne)) = n$ for $n \in \mathbb{N}$. Therefore taking first any x and y as suggested note that

$$1, x, x^2, x^3, y, y^2, xy$$

are all in $\mathcal{O}(6e)(E)$ which is of dimension 6 as a k vector space. They are 7 of them, so there is a relation with every coefficient being non-zero, that we can assume of the form

$$y^2 + p(x)y = q(x)$$

where $p(x) = \alpha + \beta x$ is a polynomnial of degree 1 and $q(x)$ a polynonial of degree 3 with coeff. in k. We can suppose that the leading coefficient y^2 is 1 up to multiplying y by the inverse of square root of it's coefficient, because k is algebraically closed. Completing the square, meaning that doing the change of variable

$$Y = y + \frac{\alpha + \beta x}{2}$$

which stays in $H^0(E,\mathcal{O}(3e)) \setminus H^0(E,\mathcal{O}(2e))$, we can suppose that $p(x)=0$. So we can suppose that we have an equation of the form

$$y^2 = q(x).$$

Because k is algebraically closed, we can suppose that $q(x)$ is monic: multiply by the inverse of a third square root of the leading coefficient. Now if σ is term in x^2 note that

$$X - \frac{\sigma}{3}$$

is a change of variable which eliminates x^2, and note that this element is still in $x \in H^0(E,\mathcal{O}(2e)) \setminus k$. Now we have found x, y such that

$$y^2 = x^3 + ax + b.$$

(3) Using that $\mathcal{O}(3e)$ is k-very ample, with the help of the Lemma at the end of document, the basis $x, y, 1$ of $\mathcal{O}(3e)(E)$ defines a closed immersion to $\mathbb{P}^2_k$. Now E is seen as an irreducible codimension closed subscheme of $\mathbb{P}^2_k$. Therefore there is some $G \in k[X,Y,Z]$ with $V_+(G)$ being exactly the image of E. But because E is of genus 1, the degree of G has to be three.

Note also that $x, y \in \mathcal{O}(3e)$ define a map

$$(x,y)\colon E \setminus e \to \mathbb{A}^2_k$$

which factors through $y^2 = x^3+ax+b$ by the above. This map is the restriction to $E \setminus e$ of the above map. Therefore the above map has to send e to the closure of this equation in $\mathbb{P}^2_k$ which is $V_+(Y^2Z = X^3 + aXZ^2 + bZ^3)$. So $V_+(G) \subset V_+(Y^2Z = X^3 + aXZ^2 + bZ^3)$ and therefore

$$(Y^2Z - (X^3 + aXZ^2 + bZ^3)) \subset G$$

because G is prime. By the degrees, we conclude that this inclusion is in fact an equality, concluding.

Removing $D_+(X)$ to the closure of $y^2 = x^3 + ax + b$ in $\mathbb{A}^2_k$ gives

$$V_+(X^3, Z) = V_+(X, Z)$$

topologically, which implies that the image of e has to be $[0:1:0]$.

□

Exercise 5. *Cohomology and affine maps.* Let E be an elliptic curve embedded in $\mathbb{P}^2_k$ as in Exercise 4. Consider the partially defined projection $\mathbb{P}^2_k \dashrightarrow \mathbb{P}^1_k$ on the first two components. Show that there is a unique induced map $f\colon E \to \mathbb{P}^1_k$ and compute the cohomology of $f_*\mathcal{O}(ne)$ for $n \in \mathbb{Z}$.

Solution key. We can extend the map into a dominant map of proper smooth curve over k so which is affine by Exercise 3.(6). So using that for affine maps

$$\mathrm{H}^i(E,\mathcal{O}(ne)) = \mathrm{H}^i(\mathbb{P}^1_k, f_*\mathcal{O}(nE)).$$

So it amounts to computing the cohomology of $\mathcal{O}(ne)$ on E. By the lemma at the end, point (5) we have for $n \geq 0$ that $h^0(E,\mathcal{O}(ne)) = n$ and

$h^1(E, \mathcal{O}(ne)) = 0$. Using Serre duality we get that that for $n \leq 0$ we have $h^1(E, \mathcal{O}(ne)) = n$ and $h^0(E, \mathcal{O}(ne)) = 0$.

□

Exercise 6. *Projection formula.* Let $f\colon X \to Y$ be a morphism of ringed spaces and let $\mathcal{E}$ be a locally free sheaf of finite rank on Y. Let $\mathcal{F}$ be any sheaf of $\mathcal{O}_X$-module. Show that there is a natural isomorphism

$$R^i f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong R^i f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

Solution key. Take an injective resolution of $\mathcal{F}$ by injective $\mathcal{O}_Y$-modules. Note that tensoring by $\mathcal{E}$ an injective sheaf gives still an injective sheaf. Indeed if we want to extend a map $\mathcal{F}_1 \to \mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{E}$ through an injection $\mathcal{F}_1 \to \mathcal{F}_2$, tensor by $\mathcal{E}^\vee$, use thar $\mathcal{I}$ is injective, and tensor back by $\mathcal{E}$. Note also that tensoring by $\mathcal{E}$ is exact. Now use the projection formula for the pushforward on the resolution to conclude.

□

Exercise 7. *Hodge numbers of projective space.* Let k be a field, and X be proper k-scheme. The *Hodge numbers of* X are defined as

$$h_{p,q}(X) := \dim_k \left(H^q \left(X, \Omega^p_{X|k} \right) \right).$$

These are important invariants of X from the view point of algebraic geometry. Using the Euler sequence and the remark below, show that for $0 \leq p, q \leq n$

$$h_{p,q}(\mathbb{P}^n_k) = \begin{cases} 1 \text{ if } p = q \\ 0 \text{ otherwise.} \end{cases}$$

Remark. Let

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

an exact sequence of finite locally free sheaves on a scheme X. Then for any $n \in \mathbb{N}$, there is an induced filtration

$$\bigwedge^n \mathcal{E} = F^0 \supset F^1 \supset \cdots \supset F^n \supset F^{n+1} = 0.$$

such that for every $0 \leq i \leq n$ we have an induced exact sequence

$$0 \to F^{i+1} \to F^i \to \bigwedge^i \mathcal{E}' \otimes \bigwedge^{n-i} \mathcal{E}'' \to 0.$$

Extra exercise. *Integral and finite maps.* Let $A \to B$ be a ring map. Suppose that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is universally closed. The goal is to show that $A \to B$ is integral. (The converse holds using the going-down theorem.)

(1) Show that $A \to B$ is integral if and only for every $b \in B$ the kernel J_b of the composition

$$A[t] \to B[t] \xrightarrow{\mathrm{ev}_{b^{-1}}} B_b$$

contains a polynomial with constant coefficient 1.

(2) Show that J_b contains a polynomial with constant coefficient 1 if and only if $\mathrm{Spec}(A[t]/(J_b+(t)))$ is empty.

(3) Using that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is universally closed, show that the induced map $\mathrm{Spec}(B_b) \to \mathrm{Spec}(A[t]/J_b)$ from item (1) is surjective. *Hint: This is the image of* $V(bt-1)$ *under the map* $\mathrm{Spec}(B[t]) \to \mathrm{Spec}(A[t])$.

(4) Argue that the squares in the diagram below are pullbacks and conclude.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \mathrm{Spec}(A[t]/(J_b+(t))) \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d, "0"] \\
\mathrm{Spec}(B_b) \arrow[r] & \mathrm{Spec}(A[t]/J_b) \arrow[r] & \mathrm{Spec}(A[t])
\end{tikzcd}$$

Now deduce that: *a morphism of schemes* $X \to Y$ *is finite if and only if it is affine and proper.*

Solution key. (1) The map $A \to B$ being integral means that every $b \in B$ is the zero of a monic polynomial in $A[t]$. Note that this happens if and only $\frac{1}{b}$ in B_b vanishes on a polynomial with constant coefficient 1, up to multiplying by sufficiently great power of b.

(2) Note that the above holds if and only if $1 \in J_b + (t)$.

(3) Note that $\mathrm{Spec}(B[t]) \to \mathrm{Spec}(A[t])$ is closed. Note also that $\mathrm{Spec}(B_b)$ can be seen as a closed subscheme of $B[t]$ by the evaluation at b^{-1} (the closed subscheme being $V(bt-1)$). Therefore as $\mathrm{Spec}(B_b) \to \mathrm{Spec}(A[t])$ of the first point is surjective onto the schematic image, being $\mathrm{Spec}(A[t]/J_b)$.

(4) The right square is clearly a pullback diagram. Note that the big diagram is indeed an empty pullback because $1 \in (bt-1, t)$. Now using the logic of pullback it implies that the left square is also a pullback. Because surjective maps pullback to surjective maps, it implies that $\emptyset = \mathrm{Spec}(A[t]/(t+J_b))$. This concludes that $A \to B$ is integral.

As a consequence if a map is affine and proper, this translates to affine and integrally closed of finite type, implying finite.

□

Here is a useful toolbox for line bundle on curves.

Lemma Let C be a regular and integral curve proper over an algebraic closed field k. Let D be an effective divisor, $\iota \colon D \to C$ the inclusion, $\mathcal{L}$ a line bundle.

(1) Consider the exact sequence
$$0 \to \mathcal{L}(-D) \to \mathcal{L} \to \iota_*\mathcal{O}_D \to 0.$$
Then $s \in \mathrm{H}^0(C,\mathcal{L})$ is sent to zero in $\mathrm{H}^0(D,\mathcal{O}_D) = \mathcal{O}_D(D)$ if and only if the Cartier divisor defined by s contains D.

(2) The line bundle is globally generated if for every $P \in C(k)$ the map $\mathcal{L}(-P) \to \mathcal{L}$ is not surjective on H^0. In other words $h^0(\mathcal{L}) = h^0(\mathcal{L}(-P)) + 1$.

(3) The line bundle $\mathcal{L}$ is k-very ample if for every point $P \in C(k)$, the line bundle $\mathcal{L}(-P)$ is globally generated.

(4) If $\deg(\mathcal{L}) < 0$ then $h^0(\mathcal{L}) = 0$.
(5) If $\deg(\mathcal{L}) \geq 2g - 1$ then $h^1(\mathcal{L}) = 0$. So $h^0(\mathcal{L}) = \deg(\mathcal{L}) + g - 1$.
(6) If $\deg(\mathcal{L}) \geq 2g$ then $\mathcal{L}$ is globally generated.
(7) If $\deg(\mathcal{L}) \geq 2g + 1$ then $\mathcal{L}$ is k-very ample.

Proof. (1) From the exact sequence, it means that $s \in \mathrm{H}^0(C, \mathcal{L}(-D)) = \mathcal{O}(-D) \cdot \mathcal{L}(C)$. Therefore when taking any trivialization say on an open affine U, the element s is sent in the ideal of $D \cap U$, which shows the claim.

(2) Follows from the first point. Indeed, being surjective would mean being an isomorphism so that the map $\mathcal{L} \to \iota_* \mathcal{O}_P$ would be zero on H^0, which happens if and only if P is in the base locus of $\mathcal{L}$.

(3) Note that if $\mathcal{L}(-P)$ is globally generated for some point $P \in C(k)$, then so is $\mathcal{L}$. Then $\mathcal{L}$ defines a morphism

$$f \colon C \to \mathbb{P}^n_k$$

for $s_0, \ldots, s_n \in \mathcal{L}(C)$ a basis. Because C is proper over k, as is $\mathbb{P}^n_k$ any map between those is proper[7] So the map is closed. That for every point P, the line bundle $\mathcal{L}(-P)$ is globally generated gives that for every other points Q there is s_i with $s_i(Q) \neq 0$, but $s_i(P) = 0$: indeed use (2) with $\mathcal{L}(-P)$. Therefore, the map f is injective, and closed, so an homeomorphism onto it's image. Now we are left to prove that $\mathcal{O}_{\mathbb{P}^n_k} \to f_* \mathcal{O}_C$ is surjective. For this, take any point $P \in C(k)$. It suffices to prove that

$$\mathcal{O}_{\mathbb{P}^n_k, f(P)} \to \mathcal{O}_{C,P}$$

is surjective. Recall that this map is determined as follows: say $s_i(P) \neq 0$. Then $x_j / x_i \in \mathcal{O}_{\mathbb{P}^n_k, f(P)}$ are sent to s_j / s_i which is the image of s_j along the inverse of $\mathcal{O}_{C,P} \xrightarrow{\cdot s_i} \mathcal{L}_P$. Because $\mathcal{L}(-P)$ is globally generated, there is some s_k with image in $\mathcal{L}_P$ not in $\mathfrak{m}_P \mathcal{L}_P$, but the image is in $\mathfrak{m}_P^2 \mathcal{L}_P$. Because $\mathfrak{m}_P / \mathfrak{m}_P^2$ is a k-vector space of dimension 1, we deduce that the map

$$\mathcal{O}_{\mathbb{P}^n_k, f(P)} / \mathfrak{m}_{f(P)}^2 \to \mathcal{O}_{C,P} / \mathfrak{m}_P^2$$

is surjective, implying that the map $\mathcal{O}_{\mathbb{P}^n_k, f(P)} \to \mathcal{O}_{C,P}$ is surjective by Nakayama. Indeed because f is proper $f_* \mathcal{O}_C$ is a coherent sheaf, and therefore $\mathcal{O}_{\mathbb{P}^n_k, f(P)} \to \mathcal{O}_{C,P}$ is a finite map. Therefore we can use Nakayama to have that $\mathfrak{m}_{f(P)} \mathcal{O}_{C,P} = \mathfrak{m}_P$. Now surjectivity follows because $\mathcal{O}_{C,P} = k \oplus \mathfrak{m}_P$ as k-vector spaces.

(4) If $h^0(\mathcal{L}) > 0$ then there is an effective Cartier linearly equivalent to $\mathcal{L}$. So $\deg(\mathcal{L}) \geq 0$.

(5) Let $\mathcal{L}$ be a line bundle of degree $d \geq 2g - 1$. Then $h^1(\mathcal{L}) = 0$. Indeed by Serre duality $h^1(\mathcal{L}) = h^0(\omega_C \otimes \mathcal{L}^\vee)$. But as by Serre duality and Riemman-Roch $\deg(\omega_C) = 2g - 2$, we have

$$\deg(\omega_C \otimes \mathcal{L}^\vee) = 2g - 2 - d < 0.$$

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[7] If X is proper over k and Y is separated over k, and $f \colon X \to Y$ is a k-map, factor f with it's graph $X \to X \times_k Y \to Y$. The first map is a closed immersion and the second is proper. So f is proper.

Therefore $h^1(\mathcal{L}) = h^0(\omega_C \otimes \mathcal{L}^\vee) = 0$. So by Riemman-Roch

$$h^0(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g.$$

(6-7) The formula in (5) holds for $\mathcal{L}$ and $\mathcal{L}(-P)$ under (6), so we see that (2) is met. The reasoning is the same for (7).

□

Remark. Note that in particular this shows that any proper integral curve over k admits a k-very ample invertible sheaf, so that any such variety is actually projective. This simplifies, according to taste, Exercise 1, week 7.