Dr. Domenico Valloni
Dr. Alapan Mukhopadhyay
Léo Navarro Chafloque

EPFL, fall semester 2025
AG II - Schemes and sheaves

Hand-in 4

Exercise to hand in. *Resolving singularities of a surface.* (Due Wednesday November 19, 12:00) Please write your solution in TEX.
Let k be a field of characteristic different than 2. Consider

$$S = \operatorname{Spec}\left(\frac{k[x,y,z]}{(x^2+y^2+z^2)}\right).$$

(1) Show that S is not smooth using the Jacobian criterion.
(2) Consider $S' \to S$ the blow-up of S at $V(x,y,z)$. Using the Jacobian criterion on each standard blow-up affine chart[1] show that S' is smooth over k.
(3) On each standard blow-up affine chart $U \subset S'$, show that

$$\operatorname{Der}_k(\mathcal{O}_{S'}(U), \mathcal{O}_{S'}(U))$$

is a free $\mathcal{O}_{S'}(U)$-module of rank 2, and find two explicit derivations which are generators of this free module.

Solution key. (1) The Jacobian criterion amounts to look at the ideal

$$(2x, 2y, 2z, x^2, y^2, z^2) = (x,y,z) \neq (1)$$

so S is not smooth over k.

(2) We use the description explained in the blow-up document on moodle. By symmetry we only consider the chart $k[x, \frac{y}{x}, \frac{z}{x}]$ of the blow-up. The strict transform in this chart will be given by

$$\frac{k\left[x, \frac{y}{x}, \frac{z}{x}\right]}{1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2}.$$

Let us write $y' = \frac{y}{z}$ and $z' = \frac{z}{x}$. The Jacobian criterion now is

$$(2y', 2z', 1 + y'^2 + z'^2) = (1)$$

which concludes to the smoothness over k.

(3) By symmetry, we only consider the chart $P := k[x, \frac{y}{x}, \frac{z}{x}]$ and write by abuse of notation "$y = \frac{y}{z}$" and "$z = \frac{z}{x}$". Let

$$A = \frac{k[x,y,z]}{1 + y^2 + z^2}.$$

We want to compute $\operatorname{Der}_k(A, A)$. Using the same method as in hand-in 3, we see that we want to look at derivations in $D \in \operatorname{Der}_k(P, A)$ such that

(1) $$D(1 + y^2 + z^2) = 0.$$

[1] Take the symmetry of the problem to your advantage.

Write

$$D = f_x \frac{\partial}{\partial x} + f_y \frac{\partial}{\partial y} + f_z \frac{\partial}{\partial z}$$

for $(f_x, f_y, f_z) \in A^3$. Using Equation (1) we get that

$$f_y 2y + f_z 2z = 0$$

Because we suppose that 2 is invertible, we get

$$f_y y + f_z z = 0.$$

Note that $D(y)$ and $D(z)$ cover $\mathrm{Spec}(A)$ because

$$(y, z, 1 + y^2 + z^2) = k[x, y, z].$$

So as in hand-in 3, we can deduce that there is some $g \in A$ with

$$D = f_x \frac{\partial}{\partial x} + g(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}).$$

We want to show that there is no relation between the generators $(\frac{\partial}{\partial x}, z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z})$. To do this we need to show that if $a_1, a_2 \in A$ with

$$a_1 \frac{\partial}{\partial x} + a_2(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}) = 0$$

then $a_1 = a_2 = 0$.[2] Evaluate this derivation at x. Then $a_1 = 0$. Evaluate this derivation at y. Then

$$a_2 z = 0$$

But z is a non-zero divisor in A. This is equivalent to say that z does not divide the polynomial $1 + y^2 + z^2$ in $k[x, y, z]$, which is true. So $a_2 = 0$, concluding. □

[2]We are exactly showing that $A \oplus A \to A\frac{\partial}{\partial x} + A(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}) = \mathrm{Der}_k(A, A) \subset \mathrm{Der}_k(P, A)$ is an isomorphism (so injective because surjective by definition), or in other words that this module is free of rank 2. This is what was asked in the statement of the exercise.