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AG II - Schemes and sheaves

Hand-in 3

Exercise to hand in. *Trivialization of the tangent bundle of an elliptic curve.* (Due Wednesday November 5, 12:00) Please write your solution in TEX.

(1) Let R be a ring, $P = R[x_1, \dots, x_n]$ and $P \to A$ a surjection, with kernel I. Show that there is an exact sequence (understand and make explicit the first map in the claimed sequence)

$$0 \to \mathrm{Der}_R(A, A) \to \bigoplus_{i=1}^{n} A \frac{\partial}{\partial x_i} \to \mathrm{Hom}_A(I/I^2, A)$$

where the last map sends $\frac{\partial}{\partial x_i}$ to the map induced by the composition

$$\frac{\partial}{\partial x_i} : I \to R[x_1, \dots, x_n] \to A$$

Why does it pass to the quotient by I^2?

We denote by $\mathrm{T}^1_{A|R} = \mathrm{Der}_R(A, A)$, the *$A$-module of R- derivations of A*. We also call this module the *tangent bundle of A over R.*

(2) Let

$$E = \mathrm{Proj}\left(\frac{\mathbb{C}[X, Y, Z]}{(Y^2 Z - (X^3 + Z^3))}\right).$$

Denote by x, y the images of $\frac{X}{Z}, \frac{Y}{Z}$ in $A_Z := \mathcal{O}_E(D_+(Z))$ and s, t the images of $\frac{X}{Y}, \frac{Z}{Y}$ in $A_Y := \mathcal{O}_E(D_+(Y))$. Show using the exact sequence from above that (meaning that any derivation is a scalar multiplication of the written generator)

$$\mathrm{T}^1_{A_Z|\mathbb{C}} = A_Z(2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y}) \quad \mathrm{T}^1_{A_Y|\mathbb{C}} = A_Y((3t^2 - 1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}).$$

(3) Moreover show that the generators displayed above agree on the intersection $D_+(YZ)$, giving a non-vanishing global section π of $\mathrm{T}_{E|\mathbb{C}}$ the sheaf defined by gluing by $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(U), \mathcal{O}(U))$ for any open affine U[1] implying that

$$\mathrm{T}_{E|\mathbb{C}} = \mathcal{O}_{E|\mathbb{C}}\pi.$$

Solution key. (1) First we note the following.

$$\mathrm{Der}_R(P, A)$$

is free A-module of rank n on the basis given by the R-derivations

$$P \xrightarrow{\frac{\partial}{\partial x_i}} P \to A.$$

[1]You can assume that this well defined so that this sheaf indeed glues, so you only need to verify that these derivations agree on the intersection.

Indeed, by the Leibniz rule, we see that an R-derivation $D \in \mathrm{Der}_R(P, A)$ is entirely determined by it's value on x_i, namely

$$D = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$$

where f_i is the value of D at x_i. For the freeness, it suffices to show that if

$$\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} = 0$$

then $f_i = 0$ for every i. But this is true but evaluating D at x_i. We therefore understand the sequence in statement as follows

$$0 \to \mathrm{Der}_R(A, A) \to \mathrm{Der}_R(P, A) \to \mathrm{Hom}_P(I, A) = \mathrm{Hom}_A(I/I^2, A)$$

where the first map is given by precomposition by the quotient map $P \to A$, and the second map is given by restriction of a derivation to I. Note that this restriction is P-linear as claimed in the sequence because of the Leibniz rule: for any $p \in P$ and $i \in I$

$$D(pi) = pD(i) + iD(p) = pD(i)$$

because $i = 0$ in A. The equality $\mathrm{Hom}_P(I, A) = \mathrm{Hom}_A(I/I^2, A)$ is the tensor adjunction between P-modules and A-modules from the surjection $P \to A$.[2]

We now show that this sequence is exact. The first map is injective because $P \to A$ is surjective. Say now that we have $D \in \mathrm{Der}_R(P, A)$ which is zero when restricting to I. By the universal property of the quotient in R-modules, we get a unique map $\overline{D}\colon A \to A$. It remains to show that it respects the Leibniz rule. But this follows because D does.

(2) By the above, we want to understand

$$D \in \mathrm{Der}_{\mathbb{C}}\left(\mathbb{C}[x, y], \frac{\mathbb{C}[x, y]}{y^2 - (x^3 + 1)}\right)$$

such that $D(y^2 - (x^3 + 1)) = 0$. Because

$$D = f_x \frac{\partial}{\partial x} + f_y \frac{\partial}{\partial y}$$

we see that this consists of derivations with

$$2y f_y = 3x^2 f_x$$

in A_Z. Note that $3x^2 A_Z + 2y A_Z = A_Z$. Indeed[3]

$$(3x^2, 2y, y^2 - (x^3 + 1)) = (1).$$

Therefore, $3x^2$ is a unit in $\mathbb{C}[x]/(x^3 + 1) = A_Z/yA_Z$, so we see that $y \mid f_x$. We therefore deduce, because $y \neq 0$ in A_Z that

$$f_y = 3x^2 g$$

[2]Note that one can directly verify that the map factors through I^2 by the Leibniz rule

$$D(ij) = iD(j) + jD(i) = 0$$

because $i, j = 0$ in A.

[3]This is the Jacobian criterion for hypersurfaces.

for some $g \in A_Z$. But now we can deduce, inserting the expression of f_y in the above, and because $3x^2$ is not zero in A_Z, that

$$2yg = f_x.$$

In other words, we have shown that $D = g(2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y})$, as desired.

For the other calculation, using the same method we end up computing pairs $(f_s, f_t) \in A_Y^2$ such that

$$f_s 3s^2 = f_t(1 - 3t^2)$$

in A_Y. We expose the same argument but in a different form. Note that $3s^2 A_Y + (1 - 3t^2)A_Y = A_Y$, because the ideal

$$(3s^2, 1 - 3t^2, t - s^3 - t^3) = (s^2, t^2 - \frac{1}{3}, t) = (1)$$

in $\mathbb{C}[s, t]$. So we have some $a, b \in A_Y$ with

$$a3s^2 = b(1 - 3t^2) + 1.$$

So multiplying by a the equation above gets that

$$f_s = (f_t - f_s b)(1 - 3t^2)$$

So inserting back in the equation above and simplifying by $1 - 3t^2$ we get

$$f_t = (f_t - f_s b)3s^2.$$

This concludes.

(3) It suffices to show that both derivations agree on the intersection. Note that $x = st^{-1}$ and $y = t^{-1}$. Now,

$$\left((3t^2 - 1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}\right)(st^{-1}) = \frac{3t^2 - 1}{t} + \frac{3s^3}{t^2}.$$

But this equals, because $s^3 = t - t^3$, to $2/t = 2y$. Also

$$\left((3t^2 - 1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}\right)(t^{-1}) = \frac{3s^2}{t^2} = 3x^2,$$

which concludes that derivations indeed agree on the intersection.

We claim that $T^1_{E|k}$ is an invertible sheaf. Indeed by (2), we need to show that the map sending 1 to the generator

$$A_Z \to A_Z(2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y})$$

is injective (because it is by definition surjective). This means that we need to show that for $a \in A_Z$ if

$$a(2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y}) = 0$$

then $a = 0$. We evaluate this derivation at x. We then get

$$a2y = 0.$$

But $2y$ is a non-zero divisor in A_Z. So $a = 0$ as desired. A totally similar argument shows that

$$A_Y \to A_Y((3t^2 - 1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t})$$

is an isomorphism.

Therefore the above shows that the generator glues to a nowhere vanishing global section π, and so

$$\mathrm{T}_{E|\mathbb{C}} = \mathcal{O}_{E|\mathbb{C}}\pi.$$

□

Remark. Note the following very general fact which we used in (2), which may look only true for UFDs, but in fact is true in any context. Say R is a ring $a, b \in R$ such that $(a, b) = 1$ which means that $aR + bR = R$. Then if $b \mid xa$, it implies that $b \mid x$. A proof: let $u, v \in R$ such that

$$au + bv = 1$$

If $b \mid xa$ this means that there is some y with $xa = by$. Multiplying by u gets

$$x = b(vx + y).$$