

26.11.25 Lecture continued.

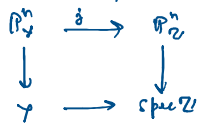
We are interested (a general version) of the following question.

Question: Let k be a field. X be a finite type scheme / k . When is X quasi-projective, i.e. when is X isom to an open subscheme of a closed subscheme of \mathbb{P}_k^n .

Notation and Terminology:

$\mathbb{P}_k^n = \text{Proj}(\mathbb{Z}[x_0, \dots, x_n])$, For any scheme Y

$\mathbb{P}_Y^n = \mathbb{P}_{\mathbb{Z}}^n \times_{\text{Spec } \mathbb{Z}} Y$, $\mathcal{O}_{\mathbb{P}_Y^n}(1) = j^* \mathcal{O}_{\mathbb{P}_k^n}(1)$.



$\mathbb{P}_{\text{Spec } A}^n =: \mathbb{P}_A^n$,

\mathcal{L} invertible sheaf on X , $s \in \Gamma(X, \mathcal{L})$.

$D_s = \{x \in X \mid s_x \neq 0\}$. $D_s \subseteq X$ open.

Note: The map $\mathcal{O}_X \rightarrow \mathcal{L}$
 $1 \mapsto s$
is an isom on D_s .

Def: For any $t \in \Gamma(U, \mathcal{L})$, $V \subseteq D_s$, $t|_V \in \mathcal{O}_X(V)$ is the unique elt $s|_V$ $t|_V = t|_s \cdot s|_V$

f Morphism to \mathbb{P}^n

Let A be a ring.

Thm: Let X be an A -scheme (i.e. fix a morphism $X \rightarrow \text{Spec } A$)

\mathcal{L} be an invertible \mathcal{O}_X -mod. $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$
s.t. $X = \bigcup_{i=0}^n D_{s_i}$.

- There is a A -scheme morphism $\varphi: X \rightarrow \mathbb{P}_A^n$ and an isom $\theta: \varphi^*(\mathcal{O}_{\mathbb{P}_A^n}(1)) \xrightarrow{\sim} \mathcal{L}$ which sends $\varphi^*(x_i)$ to s_i .

- Any A -scheme morphism $\psi: X \rightarrow \mathbb{P}_A^n$ such that there is an isom $\psi^*(\mathcal{O}_{\mathbb{P}_A^n}(1)) \rightarrow \mathcal{L}$ sending $\psi^*(x_i)$ to s_i must coincide with φ .

Proof: Given $f: X_1 \rightarrow X_2$ scheme map, $\mathcal{F}_i \in \text{Mod}_{\mathcal{O}_{X_2}}$
 $t \in \Gamma(X_2, \mathcal{F}_i)$. Recall $f^* \mathcal{F}_i := f^{-1} \mathcal{F}_i \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}$
 t gives a section $f^{-1}(t) \in \Gamma(X_1, f^{-1}(\mathcal{F}_i))$
 $f^*(t)$ denotes the section $f^{-1}(t) \otimes 1$.

More concretely, the adjunction between f^* , f_* gives a map (image of id) $\Gamma(\mathcal{F}_i) \rightarrow \Gamma(f_* f^* \mathcal{F}_i)$, $\mathcal{F}_i \rightarrow f_* (f^* \mathcal{F}_i)$.
 $f^* t$ is the image of t under this map.

Pf of Thm.

$$\mathbb{P}_A^n = \bigcup_{i=0}^n D_{x_i}, D_{x_i} = D_+(x_i) = \text{Spec}(A[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}])$$

There is a unique A -scheme map φ_i for each i

$$X \supseteq D_{s_i} \longrightarrow D_{x_i}$$

induced by the A -alg map

$$A[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}] \longrightarrow \Gamma(D_{s_i}, \mathcal{O}_X)$$

$$x_j/x_i \longmapsto s_j/s_i$$

$$\varphi_i|_{D_{s_i} \cap D_{s_j}}: D_{s_i} \cap D_{s_j} \longrightarrow D_{x_i} \cap D_{x_j}$$

both correspond to the same A -alg map

$$A[\frac{x_0}{x_i}, \frac{x_0}{x_j}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i-1}}{x_j}, \frac{x_{i+1}}{x_i}, \frac{x_{i+1}}{x_j}] \xrightarrow{[(\frac{x_j}{x_i})^{-1}]} \Gamma(D_{s_i} \cap D_{s_j}, \mathcal{O}_X)$$

$$\parallel$$

$$A[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}] \xrightarrow{[(\frac{x_i}{x_i})^{-1}]} \Gamma(D_{s_i}, \mathcal{O}_X)$$

$$A \left[\begin{array}{c} x_0/x_i, x_1/x_i, \dots, x_{n-1}/x_i, x_n/x_i \\ \parallel \\ x_0/x_i, \dots, x_{n-1}/x_i \end{array} \right] \left[\begin{array}{c} (x_i/x_i)^{-1} \\ \vdots \\ (x_i/x_i)^{-1} \end{array} \right] \longrightarrow \Gamma(D_{x_i} \cap D_{x_j}, \mathcal{O}_X)$$

$$A \left[\begin{array}{c} x_0/x_j, \dots, x_{n-1}/x_j, x_n/x_j \\ \parallel \\ x_0/x_j, \dots, x_{n-1}/x_j \end{array} \right] \left[\begin{array}{c} (x_j/x_j)^{-1} \\ \vdots \\ (x_j/x_j)^{-1} \end{array} \right]$$

- Thus $\varphi_i|_{D_{x_i} \cap D_{x_j}} = \varphi_j|_{D_{x_i} \cap D_{x_j}}$
- Thus $\exists!$ $\varphi: X \rightarrow \mathbb{P}_A^n$ such that $\varphi|_{D_{x_i}} = \varphi_i$.

The desired isom $\varphi^*(\mathcal{O}_{\mathbb{P}_A^n}(1)) \rightarrow \mathcal{L}$ is constructed as follows. Note $\varphi^{-1}(D_{x_i}) = D_{x_i}$ by construction.

$$\begin{array}{ccc} \theta_i: \varphi^*(\mathcal{O}_{\mathbb{P}_A^n}(1))|_{D_{x_i}} & \xrightarrow{\sim} & \mathcal{O}_{D_{x_i}} \cdot \varphi^*(x_i) \longrightarrow \mathcal{O}_{D_{x_i}} \cdot s_i \xrightarrow{\sim} \mathcal{L}|_{D_{x_i}} \\ \downarrow \cong & & \downarrow \cong \\ \varphi_i^*(\mathcal{O}_{\mathbb{P}_A^n}(1)|_{D_{x_i}}) & \xrightarrow{\sim} & \varphi_i^*(\mathcal{O}_{\mathbb{P}_A^n}(1)|_{D_{x_i}} \cdot x_i) \end{array}$$

Since $\theta_i|_{D_{x_i} \cap D_{x_j}} = \theta_j|_{D_{x_i} \cap D_{x_j}}$
 $\exists!$ $\theta: \varphi^*(\mathcal{O}_{\mathbb{P}_A^n}(1)) \rightarrow \mathcal{L}$ s.t. $\theta|_{D_{x_i}} = \theta_i$

$$\text{Clearly } \theta(\varphi^*(x_i)) = s_i \quad \forall i, \quad (\theta_j(\varphi^*(x_i)) = \theta_j(\frac{x_i}{x_j} \varphi^*(x_j)) = \frac{x_i}{x_j} s_j = s_i)$$

For the uniqueness, given such a ψ .

The existence of the isom $\varphi^*(\mathcal{O}_{\mathbb{P}_A^n}(1)) \rightarrow \mathcal{L}$ sending $\varphi^*(x_i)$ to s_i implies $\psi^{-1}(D_{x_i}) = D_{x_i}$. Indeed for $x \in X$, we have an isom

$$\begin{array}{ccc} \mathbb{k}(x) & \otimes & \frac{\mathcal{O}_{\mathbb{P}_A^n}(1)_{\psi(x)}}{\mathcal{O}_{\mathbb{P}_A^n}(1)_{\psi(x)}} \xrightarrow{\sim} \frac{\mathcal{L}_x}{\mathcal{L}_x} \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{k}(x) & \otimes & \mathbb{k}(x) \end{array}$$

$$x \in X - D_{x_i} \Leftrightarrow s_i \pmod{\mathcal{L}_x} = 0 \Leftrightarrow 1 \otimes x_i \pmod{\mathcal{O}_{\mathbb{P}_A^n}(1)_{\psi(x)}} = 0$$

Since $\mathbb{k}(x) \hookrightarrow \mathbb{k}(\psi(x)) \subset 0$ field extension the last quantity is zero iff $x_i \pmod{\mathcal{O}_{\mathbb{P}_A^n}(1)} = 0 \Leftrightarrow \psi(x) \in \mathbb{P}_A^n - D_{x_i}$

That map $A[x_0/x_i, \dots, x_n/x_i] \rightarrow \Gamma(D_{x_i}, \mathcal{O}_X)$
 That induces $\psi|_{D_{x_i}}$ sends x_j/x_i to s_j/s_i :

$$\begin{aligned} \text{On } D_{x_i} \quad \psi^*(x_j) &= \psi^*(x_j/x_i) \cdot \psi^*(x_i) \\ \text{So } \psi^*(x_j/x_i) &= \psi^*(x_j) / \psi^*(x_i) \\ &= s_j/s_i \end{aligned}$$

Thus $\psi|_{D_{x_i}} = \varphi|_{D_{x_i}} \quad \forall i$. End of 22.11.24 lecture

Prop When X is f.t. over an algebraically closed field.

The map induced by s_0, \dots, s_n sends a closed pt $x \in X$ to $[s_0(x) : \dots : s_n(x)] \in \mathbb{P}_A^n$.
 Given an A -scheme X and $s_i \in \Gamma(X, \mathcal{L})$.

Thm \mathbb{A}^1 There is a one to one correspondence
 $\{ \mathcal{L}, \text{ordered tuple } (s_0, \dots, s_n) \mid \mathcal{L} \text{ invertible, } s_i \in \Gamma(X, \mathcal{L}) \text{ s.t. } \bigcup_{i=0}^n D_{s_i} = X \}$ \longleftrightarrow $\{ A\text{-scheme map } X \rightarrow \mathbb{P}_A^n \}$

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where $(\mathcal{L}_1, (s_0, \dots, s_n)) \sim (\mathcal{L}_2, (t_0, \dots, t_n))$
 iff \exists an isom of \mathcal{O}_X -mods $\theta: \mathcal{L}_1 \rightarrow \mathcal{L}_2$ sending s_i to $t_i \quad \forall i$.

Pf Given $\varphi: X \rightarrow \mathbb{P}_A^n$ of A -schemes take $\mathcal{L}_\varphi = \varphi^*(\mathcal{O}_{\mathbb{P}_A^n}(1))$ and $s_i = \varphi^*(x_i)$.

The morphism induced by $(\mathcal{L}_\varphi, (s_0, \dots, s_n))$ is indeed φ . The one-to-one correspondence is H.H.

Eg: $\mathbb{k} = \mathbb{C}, \mathbb{Q}, \dots \quad X = \mathbb{P}_{\mathbb{k}}^1, \mathcal{L} = \mathcal{O}(2)$
 $\Gamma(\mathbb{P}_{\mathbb{k}}^1, \mathcal{O}(2)) = \mathbb{k}x^2 \oplus \mathbb{k}xy \oplus \mathbb{k}y^2$.

$$\text{get } \mathbb{P}_{\mathbb{k}}^1 \longrightarrow \mathbb{P}_{\mathbb{k}}^2$$

$$[x_1 : x_0] \longrightarrow [x_0^2 : x_1 x_0 : x_1^2]$$

The image is $V(Y^2 - XZ)$

End of 26.11.25 lecture

§ Criterion for having an embedding

Recall: A locally closed subspace of a topological space is an intersection of a closed and open subset with the induced topology.

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Def: A morphism of schemes $\varphi: X \rightarrow Y$ is called a locally closed immersion if (1) φ induces a homeomorphism X to a locally closed subset of Y and (2) at all points of $y \in \varphi(X)$ $\mathcal{O}_{Y,y} \rightarrow (\varphi_* \mathcal{O}_X)_y$ is surjective.] Locally closed immersion

$\varphi: X \rightarrow Y$ is called a closed immersion if $\varphi(X) \subseteq Y$ is closed.

$\varphi: X \rightarrow Y$ is called an open immersion if $\varphi(X) \subseteq Y$ is open and $\forall y \in \varphi(X)$, $\mathcal{O}_{Y,y} \rightarrow (\varphi_* \mathcal{O}_X)_y$ is an isom.

Prop: Every locally closed immersion $\varphi: X \rightarrow Y$ can be factored as

$X \xrightarrow{i} U \xrightarrow{j} Y$ where j is an open immersion and i is a closed immersion.

pf: Write $\varphi(X) = Z \cap U$ where $Z \subseteq Y$ is closed, $U \subseteq Y$ is open.

Then φ factors as $X \xrightarrow{i} U \xrightarrow{j} Y$ as a scheme map where U has the induced open scheme structure and j is the open immersion.

We claim that i is a closed embedding. Indeed i is a homeo onto the image and $i(X) = Z \cap U$ is closed in U .

Note that for a point in U which is not in $i(X)$, the stalk $(i_* \mathcal{O}_X)_y = \varphi_* (\mathcal{O}_X)_y = 0$. Indeed one can choose an open nbhd V of y such that $V \cap i(X) = \emptyset$. Then $i^{-1}(V) = \emptyset$. So $(i_* \mathcal{O}_X)_y = 0$.

We claim that the induced map $\mathcal{O}_U \rightarrow i_* \mathcal{O}_X$ is surjective.

If $y \in i(X)$, $\mathcal{O}_{U,y} \rightarrow (i_* \mathcal{O}_X)_y$ is the same as the induced map $\mathcal{O}_{Y,y} \rightarrow (\varphi_* \mathcal{O}_X)_y$ which is surjective by our assumption.

If $y \notin i(X)$, $\mathcal{O}_{U,y} \rightarrow (i_* \mathcal{O}_X)_y$ is surjective because the target is zero.

Remark: In general it is not true that any locally closed immersion $\varphi: X \rightarrow Y$ can be factored as

$X \xrightarrow{i} Z \xrightarrow{j} Y$ where j is an open immersion and i is a closed immersion.

However, we have the following.

Prop: Let $\varphi: X \rightarrow Y$ be a locally closed immersion such that $\varphi_* \mathcal{O}_X$ is quasi-coherent. Then $\varphi: X \rightarrow Y$ can be factored as $X \xrightarrow{j} Z \xrightarrow{i} Y$

where j is an open immersion and i is closed.

pf: Let $\mathcal{I} = \ker(\mathcal{O}_Y \rightarrow \varphi_* \mathcal{O}_X)$. For $y \in \varphi(X)$,

since $\mathcal{O}_{Y,y} \rightarrow (\varphi_* \mathcal{O}_X)_y$ is sur, $\mathcal{I}_y \subseteq \mathcal{m}_y$.

Since $\varphi_* \mathcal{O}_X$ is qcsh, \mathcal{I} is qcsh.

So $V(\mathcal{I}) = \{y \in Y \mid \mathcal{I}_y \subseteq \mathcal{m}_y\}$ is closed in Y .

$V(\mathcal{I}) \supseteq \varphi(X)$. Equip $V(\mathcal{I})$ with the scheme structure given by $\mathcal{O}_{Y/\mathcal{I}}$, call the scheme Z .

Note $\varphi(X)$ is open in $V(\mathcal{I})$. Since $\mathcal{I} \cdot \varphi_* \mathcal{O}_X = 0$,

the map $\varphi: X \rightarrow Y$ factors through the closed immersion $\varphi: X \xrightarrow{j} Z \xrightarrow{i} Y$, i is the closed immersion.

We claim j is an open immersion. Indeed what remains to check is that $\forall y \in \varphi(X) \subseteq Z$, the induced map $\mathcal{O}_{Z,y} \rightarrow (j_* \mathcal{O}_X)_y$ is an isom. But

$\mathcal{O}_{Z,y} = \mathcal{O}_{Y,y} / \mathcal{I}_y$ is isom to $(\varphi_* \mathcal{O}_X)_y$ for $y \in \varphi(X)$.

is the map induced by φ and hence by j . \square

Def: Given a scheme Y . Sch_Y is the category

whose objects are schemes X together with a morphism $X \rightarrow Y$.

Given $X_1 \rightarrow Y, X_2 \rightarrow Y$ in Sch_Y a Y -scheme

morphism is a scheme map $\varphi: X_1 \rightarrow X_2$ s.t

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_2 \\ & \searrow & \swarrow \\ & Y & \end{array}$$

commutes.

Def: Let X be a Y -scheme. An invertible \mathcal{O}_X mod \mathcal{L} is called Y -very ample if

Def. Let X be a Y scheme. An invertible \mathcal{O}_X mod \mathcal{L} is called Y -very ample if $X \rightarrow Y$ can be factored as

$$X \xrightarrow{\varphi} \mathbb{P}_Y^n \rightarrow Y$$

where φ is locally closed and $\mathcal{L} \cong \varphi^*(\mathcal{O}_{\mathbb{P}_Y^n}(1))$.

• Whenever the structure map $X \rightarrow Y$ has a factorization as in $*$, we say X is quasi-projective over Y .

• Whenever the structure map $X \rightarrow Y$ can be factored as in $*$ with φ being a closed embedding, the morphism $X \rightarrow Y$ is called projective.

Prop. A ringed set X/A be projective. Any locally closed immersion $\varphi: X \rightarrow \mathbb{P}_A^n$ is a closed immersion. Thus X is projective $/A$ $\Leftrightarrow X$ is quasi-projective $/A$.

Pf. \mathbb{P}_A^n is separated, X/A separated. So $\varphi(X)$ is closed in \mathbb{P}_A^n . \square

Thm. Let X be a finite type scheme $/A = \text{noetherian}$; \mathcal{L} be an invertible sheaf on X . \mathcal{L} is ample $\Leftrightarrow \mathcal{L}^n$ is very ample for some n .

Pl. In our setup a very ample invertible sheaf \mathcal{L}' is ample.

Indeed, choose a locally closed embedding $X \xrightarrow{\varphi} \mathbb{P}_A^n$ s.t. $\varphi^* \mathcal{O}_{\mathbb{P}_A^n}(1) \cong \mathcal{L}'$.

Factor φ as $X \xrightarrow{i} U \xrightarrow{j} \mathbb{P}_A^n$, where i is a closed immersion and

j is an open immersion. Let $F \in \text{Coh}(X)$. Since i is a closed immersion $i_* F \in \text{Coh}(U)$ and $i^*(i_* F) \cong F$. Recall $\exists G \in \text{Coh}(\mathbb{P}_A^n)$ s.t. $j^* G \cong i_* F$.

Recall that $\mathcal{O}_{\mathbb{P}_A^n}(1)$ is ample. So $\exists n_0$ s.t. $\forall n > n_0$ $G \otimes \mathcal{O}_{\mathbb{P}_A^n}(n)$ is globally generated. So $\varphi^*(G \otimes \mathcal{O}_{\mathbb{P}_A^n}(n)) \cong \varphi^*(G) \otimes \mathcal{L}^n \cong F \otimes \mathcal{L}^n$ is globally generated $\forall n > n_0$.

\mathcal{L}^m very ample $\Rightarrow \mathcal{L}^{mn}$ ample $\Rightarrow \mathcal{L}$ ample.

Now assume \mathcal{L} is ample.

Step 1: For each $x \in X$, there exists $\mathcal{L}_x \in \Gamma(X, \mathcal{L}^{n_x})$ such that $D_{\mathcal{L}_x}$ is an affine open nbhd of x .

Pl. of step 1: Fix $x \in X$ and an affine open nbhd U_x of x . Let $\mathcal{I} \in \mathcal{O}_x$ be an ideal sheaf s.t. $V(\mathcal{I}) = U_x$. Choose n_x s.t. $\mathcal{I} \otimes \mathcal{L}^{n_x}$ is globally generated. So there exists $\mathcal{L}_x \in \Gamma(U_x, \mathcal{I} \otimes \mathcal{L}^{n_x}) \cong \Gamma(U_x, \mathcal{L}^{n_x})$ s.t. $\mathcal{L}_x \notin m_x(\mathcal{I} \otimes \mathcal{L}^{n_x}) = m_x \mathcal{L}^{n_x}$ [$\because \mathcal{I}_x = \mathcal{O}_{x, n_x}$].

Note $\forall y \in U_x, (\mathcal{L}_x)_y \in \mathcal{I}_y \otimes \mathcal{L}_y^{n_x} \subseteq m_y \mathcal{L}_y^{n_x}$

So $D_{\mathcal{L}_x} \subseteq U_x$. Choose an ideal $\mathcal{I}' \in \mathcal{O}_{U_x}$. Assume \mathcal{L}_x generates \mathcal{I}' . $D_{\mathcal{L}_x} = D_{\mathcal{I}'}$ in U_x . So $D_{\mathcal{L}_x}$ is affine. \square

Since X is quasi-compact, there is a finite covering

$$X = \bigcup_{i=1}^s D_{\mathcal{L}_i}$$

Replace \mathcal{L} by $\mathcal{L}^{n_1 n_2 \dots n_s}$ and \mathcal{L}_i by \mathcal{L}_i .

Then $\exists \mathcal{L}_1, \dots, \mathcal{L}_s \in \Gamma(X, \mathcal{L})$ such that

$$X = \bigcup_{i=1}^s D_{\mathcal{L}_i} \text{ and each } D_{\mathcal{L}_i} \text{ is affine}$$

For each $i \leq s$, make a choice

$$\Gamma(D_{\mathcal{L}_i}, \mathcal{L}) = A[\mathcal{Y}_j \mid 1 \leq j \leq n_i]$$

$\exists \mathcal{L} \otimes \mathcal{L}^n$ s.t. $\mathcal{Y}_j \otimes \mathcal{L}_i$ extends to a global section \mathcal{T}_j of $\Gamma(X, \mathcal{L}^n)$, $\forall 1 \leq i \leq s, 1 \leq j \leq n_i$

Consider the morphism φ to some \mathbb{P}^N given by
 $\mathcal{L} = \{t_i\}, \mathcal{L}^{\otimes 2} = \{t_i^2\}, \dots, \mathcal{L}^{\otimes n} = \{t_i^n\}, \dots, \mathcal{L}^{\otimes n} = \{t_i^n\}, \dots$
 $N = n_1 + n_2 + \dots + n_s + n_s - 1$
 $\mathbb{P}^N = \text{Proj}(A[x_i, y_i | 1 \leq i \leq s, 1 \leq j \leq n_i])$

Claim: φ is a locally closed immersion

Pf: Since $X = \bigcup_{i=1}^s D_{+} t_i$, φ factors through $\bigcup_{i=1}^s D_{+} t_i = V$

We show $\varphi: X \rightarrow V \subseteq \mathbb{P}^N$ is a closed immersion.

For that, enough to show

$D_{+} t_i = \varphi^{-1}(D_{+} t_i) \rightarrow D_{+} t_i$ is a closed immersion $\forall i$

$\Leftrightarrow \Gamma(D_{+} t_i, \mathbb{P}^N) \rightarrow \Gamma(D_{+} t_i, X)$ is surjective $\forall i$

$$\frac{t_i}{t_i^2} \longmapsto \frac{A[t_i | 1 \leq i \leq s]}{t_i} \quad [\because t_i \text{ is invertible in } t_i \otimes \mathcal{L}_i^{\otimes 2}] \quad \square$$

Prop. (i): A noetherian ring, $X = \text{Spec } A$ is quasi-projective iff X has an ample invertible sheaf.

(ii) If X is $\text{Spec } A$, then X is projective $\Leftrightarrow X$ has an ample invertible sheaf.

Examples of projective morphisms. Blows-ups.

- X noetherian scheme, $\mathcal{Y} = \bigoplus_{n \in \mathbb{N}} \mathcal{Y}_n$ be an \mathbb{N} -graded quasi-coherent sheaf of \mathcal{O}_X -algebras such that
 - Each \mathcal{Y}_n is a q.coh \mathcal{O}_X -submod of \mathcal{Y}
 - The \mathcal{O}_X -alg structure on \mathcal{Y} , comes from a ring map $\mathcal{O}_X \rightarrow \mathcal{Y}_0$

for each affine $\text{Spec } (A) \subseteq X$
 $\mathcal{Y}(\text{Spec } A) = (S_0 \oplus S_1 \oplus S_2 \oplus \dots)^{\vee}$
 where $\oplus S_i$ is standard graded.

Proof: There exists a scheme denoted $\text{Proj}_X(\mathcal{Y})$ with a map $\pi: \text{Proj}_X(\mathcal{Y}) \rightarrow X$ such that (see page 166 Hom1.)
 for each affine open $\text{Spec } (A) \subseteq X$,
 $\pi^{-1}(U) \cong \text{Proj}(S_0 \oplus S_1 \oplus \dots \oplus S_n)$ where

$\mathcal{Y}(\text{Spec } A) = (\oplus S_i)^{\vee}$
 • There exists an invertible sheaf denoted $\mathcal{O}(1)$ on $\text{Proj}_X(\mathcal{Y})$ s.t. $\mathcal{O}(1)|_{\text{Spec } (A)} \cong S(1)$.
 where $S = \oplus S_i$.

• Given an invertible sheaf \mathcal{L} and \mathcal{Y} as above define $\mathcal{Y}' = \bigoplus \mathcal{Y}_n \otimes \mathcal{L}^n$

Then there is a natural isom $\varphi: \text{Proj}(\mathcal{Y}') \rightarrow \text{Proj}(\mathcal{Y})$ such that $\mathcal{O}(1) \cong \varphi^* \mathcal{O}(1) \otimes \pi^* \mathcal{L}$ where $\pi: \text{Proj}(\mathcal{Y}) \rightarrow X$ is the natural projection map.

eg: $\mathcal{Y} = \bigoplus_{n \in \mathbb{N}} \mathcal{O}_X(n)$ $\varphi \in \mathbb{N}$
 $\mathcal{Y} = \text{Sym}^*(\mathcal{E}) = \mathcal{O}_X \oplus \text{Sym}^1 \mathcal{O}_X \oplus \text{Sym}^2 \mathcal{O}_X \oplus \dots$
 On $\text{Spec } A$, $\mathcal{Y}(\text{Spec } A) \cong A[t_0, \dots, t_n]$
 "so" $\text{Proj}(\mathcal{Y}) = X \times_{\mathbb{A}^1} \mathbb{P}^n$.

Proof: If X has an ample invertible sheaf \mathcal{L} , and \mathcal{Y} is coherent.
 Then $\pi: \text{Proj}(\mathcal{Y}) \rightarrow X$ is projective.

Pf: Choose $n \in \mathbb{N}$ s.t. $\mathcal{Y}_i \otimes \mathcal{L}^n$ is glob gen.
 take $\mathcal{L}' = \mathcal{L}^n$.

Choose a surjection $\bigoplus_{i=1}^{r+1} \mathcal{O}_X \rightarrow \mathcal{Y}_i \otimes \mathcal{L}'$
 This gives an surjection $\text{Sym}^n(\bigoplus_{i=1}^{r+1} \mathcal{O}_X) \rightarrow \bigoplus \mathcal{Y}_i \otimes \mathcal{L}'^n = \mathcal{Y}'$
 \mathcal{O}_X -lin

and Thus a closed embedding of $\text{Spec } X$.
 $\text{Proj}(\mathcal{Y}') \rightarrow \text{Proj}(\text{Sym}^n(\bigoplus_{i=1}^{r+1} \mathcal{O}_X)) = X \times_{\mathbb{A}^1} \mathbb{P}^r$

But $\text{Proj}(\mathcal{Y}') = \text{Proj}(\mathcal{Y})$.

Def: Given an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. The blow up of X along \mathcal{I} is the X -scheme

$$\pi: \text{Proj}(\bigoplus_{n \in \mathbb{N}} \mathcal{I}^{\otimes n}) \rightarrow X, \text{ set } \text{Bl}_{\mathcal{I}}(X) = \text{Proj}(\bigoplus_{n \in \mathbb{N}} \mathcal{I}^{\otimes n})$$

Notation: Given $f: X_1 \rightarrow X_2$ and an ideal sheaf \mathcal{I} .

$$(O_X \otimes \mathcal{I}^n \otimes \mathcal{I}^n \otimes \dots) \quad \text{--- } n \in \mathbb{N}$$

Notation: Given $f: Y_1 \rightarrow Y_2$ and an ideal sheaf $\mathcal{I}_2 \subset \mathcal{O}_{Y_2}$

$f^{-1}(\mathcal{I}_2) \mathcal{O}_{Y_1} = \mathcal{I}_2 \mathcal{O}_{Y_1}$
 is the ideal sheaf obtained as the image of the map $f^*(\mathcal{I}_2) \rightarrow \mathcal{O}_{Y_1}$

Thm. Let X be a noetherian scheme

(i) Given an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, consider the blow-up map

$$\pi: \text{Bl}_{\mathcal{I}}(X) \rightarrow X$$

Then $\pi^{-1}(\mathcal{I}) \mathcal{O}_{\text{Bl}_{\mathcal{I}}(X)} \cong \mathcal{O}(-1)$

In particular $\mathcal{I} \mathcal{O}_{\text{Bl}_{\mathcal{I}}(X)}$ is invertible.

(ii) $\pi^{-1}(X \setminus V(\mathcal{I})) \rightarrow X \setminus V(\mathcal{I})$ is an isom.

(iii) Let $g: Z \rightarrow X$ be a scheme map such that $\mathcal{I} \mathcal{O}_Z$ is invertible. Then g uniquely factors as

$$\begin{array}{ccc} Z & \dashrightarrow & \text{Bl}_{\mathcal{I}}(X) \\ g \searrow & & \swarrow \pi \\ & X & \end{array}$$

End of 28.11.25 Lecture

Pf. We prove only (iii). First, we prove existence of the factorization.

w.l.o.g. X is affine. Assume $X = \text{Spec } R$ and $\mathcal{I} = \bar{I}$, where $I = \langle f_0, \dots, f_n \rangle$.

We have a surjection of graded R -algebras

$$R[x_0, \dots, x_n] \rightarrow R \oplus I \oplus I^2 \oplus \dots$$

where on the left $\deg x_i = 1$, $\deg R = 0$. A homogeneous polynomial $F(x_0, \dots, x_n)$ is mapped to zero $\Leftrightarrow F(f_0, \dots, f_n) = 0$ in R .

The above ring map gives an embedding of R -schemes

$$\varphi: \text{Bl}_{\mathcal{I}}(X) \hookrightarrow \mathbb{P}_R^n, \text{ where } \varphi^* \mathcal{O}_{\mathbb{P}_R^n}(1) \cong \pi^{-1}(\mathcal{I} \mathcal{O}_{\text{Bl}_{\mathcal{I}}(X)})$$

isom sends $\varphi^*(x_i)$ to $\pi^{-1}(f_i) \in \pi^{-1}(\mathcal{I} \mathcal{O}_{\text{Bl}_{\mathcal{I}}(X)})$.

Consider $h: Z \rightarrow \mathbb{P}_R^n$ given by $g^{-1}(f_i) \in g^{-1}(\mathcal{I} \mathcal{O}_Z)$. h is 'defined' everywhere on Z because f_i generate I and hence $g^{-1}(f_i)$ generate $g^{-1}(\mathcal{I} \mathcal{O}_Z)$. We want to argue that h factors through $\varphi(\text{Bl}_{\mathcal{I}}(X))$. In this discussion, we use the following lemma, whose proof is left as an exercise.

Lemma: Let A be a noetherian ring, $A[x_0, \dots, x_n]$ has graded str s.t $\deg(A) = 0$, $\deg(x_i) = 1$; $\mathcal{J} \subset A[x_0, \dots, x_n]$ be a homogeneous ideal. Let $Y = \text{Proj}(A[x_0, \dots, x_n]/\mathcal{J})$. An A -scheme map $\psi: S \rightarrow \mathbb{P}_A^n$ factors through $Y \Leftrightarrow$ For any homogeneous $F \in \mathcal{J}$ $\psi^*(F(x_0, \dots, x_n)) \in \mathcal{O}_{\mathbb{P}_A^n}(\deg F)(\mathbb{P}_A^n) = 0 \Leftrightarrow \psi^*(\mathcal{O}_{\mathbb{P}_A^n}(-\deg F))(S) = 0$

Back to the proof of Thm; we need to check that for any homogeneous F s.t $F(f_0, \dots, f_n) = 0 \in R$, $F(g^{-1}(f_0), \dots, g^{-1}(f_n)) = 0$ which is a tautology.

Now, we prove uniqueness. Suppose, we have a commutative diag

$F(g^{-1}(z_0)) \dots g^{-1}(z_n) = 0$ when $v = \dots$
 Now, we prove uniqueness. Suppose, we have a commutative diag

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\psi} & \mathbb{B}L_S(X) \xrightarrow{\varphi} \mathbb{P}_R^n \\ & \searrow g & \swarrow \pi \\ & & X \end{array}$$

Recall that $\varphi^*(\mathcal{O}(1)) \cong \pi^*(\mathcal{S}) \otimes_{\mathbb{B}L_S(X)} \mathcal{O}_{\mathbb{B}L_S(X)}$. Note that $\psi^*(\pi^*(\mathcal{S}) \otimes_{\mathbb{B}L_S(X)} \mathcal{O}_{\mathbb{B}L_S(X)}) \cong \psi^*(\pi^*(\mathcal{S}) \otimes_{\mathbb{B}L_S(X)} \mathcal{O}_{\mathbb{B}L_S(X)}) \xrightarrow{\sim} g^{-1}(\mathcal{S}) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Z}}$.
 as $g = \pi \cdot \psi$.
 is invertible

Now, note that $\psi: \mathbb{Z} \rightarrow \mathbb{P}_R^n$ is determined by $\psi^*(z_i) \dots \psi^*(z_n) \in \Gamma(\mathbb{Z}, \psi^*(\mathcal{O}(1)))$.

But $\psi^*(\mathcal{O}(1)) \cong g^{-1}(\mathcal{S}) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Z}}$, from the argument above. The isomorphism sends $\psi^*(z_i)$ to $g^{-1}(z_i)$. This proves uniqueness of ψ .

f Resolving indeterminacy by blowing up.

Thm: Let A be a noetherian ring, X be a $\wedge A$ -scheme, \mathcal{L} be an invertible \mathcal{O}_X -mod. Let $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ ($n \geq 1$) be non-zero sections. Let $U = \bigcup_{i=0}^n D_{s_i}$. Then \exists a coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ s.t. $V(\mathcal{I}) \subseteq X \setminus U$ and $\varphi: \mathbb{B}L_S(X) \rightarrow \mathbb{P}_A^n$ such that the

following diag commutes

$$\begin{array}{ccc} \mathbb{B}L_S(X) & \cong \pi^{-1}(X \setminus U) & \xrightarrow{\varphi} \mathbb{P}_A^n \\ & \downarrow \pi & \nearrow [s_0 : \dots : s_n] \\ & U & \end{array}$$

Pf. Note that there is a coherent ideal sheaf \mathcal{I} such that the image of

$$\begin{array}{ccc} \mathcal{O}_X^{\oplus n+1} & \longrightarrow & \mathcal{L} \longrightarrow \mathcal{O}_X \\ e_i & \longrightarrow & s_i \end{array}$$

is \mathcal{I} .

Since the above map is surjective on U , $V(\mathcal{I}) \subseteq X \setminus U$.

Let $Y = \mathbb{B}L_S(X) \xrightarrow{\pi} X$.

By pulling back \mathcal{I} , we get a surjective map of \mathcal{O}_Y -mods

$$\bigoplus_{i=0}^n \mathcal{O}_Y \longrightarrow \pi^*(\mathcal{I}) \quad e_i \longmapsto \pi^*(s_i)$$

Check that $\pi^*(\mathcal{I}) \cong \pi^*(\mathcal{I}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{L}$, thus it is invertible

i.e. The map given by $\pi^*(s_0) \dots \pi^*(s_n)$

invariable

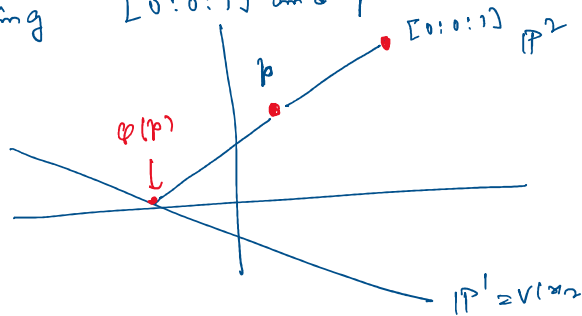
Take φ to be the map given by $\pi^0(x_0), \dots, \pi^0(x_n) \in \Gamma(Y, \pi^*(\mathcal{O}(1)))$

Commutativity of the diag follows easily.

Eg. Let k be a field. Consider the 'map' given by $x_0, x_1 \in \Gamma(\mathbb{P}_k^2, \mathcal{O}(1))$. $\Delta x_0 \cup \Delta x_1 = \mathbb{P}^2 - [0:0:1]$

So, we have a map $\varphi: \mathbb{P}^2 - [0:0:1] \rightarrow \mathbb{P}^1$
 $[x_0:x_1:x_2] \rightarrow [x_0:x_1]$

Note that $V(x_2) \cong \mathbb{P}_k^1 \subseteq \mathbb{P}^2$. Check that $\varphi: \mathbb{P}^2 - [0:0:1] \rightarrow V(x_2) = \mathbb{P}^1$ follows: For a closed pt $p \in \mathbb{P}^2 - [0:0:1]$, take the line joining $[0:0:1]$ and p . This line intersects $V(x_2)$ at $\varphi(p)$



Let $\pi: \mathbb{R} \setminus \{0\} \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be the below pt map.
 $\pi^{-1}([0:0:1])$ parametrizes the tangent directions at $[0:0:1]$.

I want to explain an intuitive reason for why adding a point for each tangent direction is necessary to extend φ .
 For each tangent direction t at $[0:0:1]$, which is a line through $[0:0:1]$ take any sequence of pts on $t - [0:0:1]$ $(x_n, t) \rightarrow [0:0:1]$. Then $\varphi(x_n, t)$ is constant. If we change t , $\varphi(x_n, t)$ changes. So any continuous extension of φ requires adding an image pt for each tangent direction.