Computation by Cech complex

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12.12.25 lecture CTD

T top space, $T = \bigcup_{i \in I} U_i$ be an open covering.

Fix an well ordering on the indexing set I. Let $\mathcal{U}$ denote the ordered collection $\{U_i\}_{i \in I}$

Given a sheaf of ab gps $\mathcal{F}$, one can associate a complex of sheaves denoted $\underline{C}^\bullet(\mathcal{U}, \mathcal{F})$ where

- For $p \geq 0$, $\underline{C}^p(\mathcal{U}, \mathcal{F})$

$$= \prod_{i_0 < i_1 < \cdots < i_p} (j_{i_0 \cdots i_p})_* \left(\mathcal{F}|_{U_{i_0 \cdots i_p}}\right)$$

where $U_{i_0 \cdots i_p} := U_{i_0} \cap U_{i_1} \cdots \cap U_{i_p}$.

and $j_{i_0 \cdots i_p} : U_{i_0 \cdots i_p} \hookrightarrow X$ is the inclusion.

- for $p < 0$ $= 0$

So for $V \underset{\text{open}}{\subseteq} T$,

$$\underline{C}^p(\mathcal{U}, \mathcal{F})(V) = \prod_{i_0 < i_1 < \cdots < i_p} \mathcal{F}(V \cap U_{i_0 \cdots i_p})$$

- For $\underline{C}^p(\mathcal{U}, \mathcal{F})(V) \ni \alpha = (\alpha_{i_0 \cdots i_p})$

$d\alpha \in C^{p+1}(\mathcal{U}, \mathcal{F})$ is given by

$$(d\alpha)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \cdots i_{j-1} \hat{i_j} i_{j+1} \cdots i_{p+1}}\Big|_{U_{i_0 \cdots i_{p+1}} \cap V}$$

<u>**Def:**</u> • $\underline{C}^\bullet(\mathcal{U}, \mathcal{F})$ is called the Čech complex of sheaves of $\mathcal{F}$ for the covering $\mathcal{U}$.

$\check{\underline{H}}^i(\mathcal{U}, \mathcal{F}) := H^i(\underline{C}^\bullet(\mathcal{U}, \mathcal{F}))$ called the i-th Čech cohomology sheaf

- $C^\bullet(\mathcal{U}, \mathcal{F}) = \Gamma(T, \underline{C}^\bullet(\mathcal{U}, \mathcal{F}))$

Čech complex of $\mathcal{F}$ for the covering $\mathcal{U}$.

$\check{H}^i(\mathcal{U}, \mathcal{F}) = H^i(\Gamma(T, \underline{C}^\bullet(\mathcal{U}, \mathcal{F})))$ is called the i-th Čech cohomology group of $\mathcal{F}$.

The i-th Čech cohomology group of $\mathcal{F}$.

Eg: $X = \mathbb{P}^1_k = \mathrm{Proj}(k[x,y])$, $\mathcal{U} = (D_+(x), D_+(y))$

$C^\bullet(\mathcal{U}, \mathcal{O}_X) =$

$$0 \to \Gamma(D_+(x), \mathcal{O}_X) \oplus \Gamma(D_+(y), \mathcal{O}_X) \longrightarrow \Gamma(D_+(xy), \mathcal{O}_X) \longrightarrow 0 \ldots$$

$$0 \to k[y/x] \oplus k[x/y] \longrightarrow k[x/y, y/x] \longrightarrow 0$$

$$(\alpha_0, \alpha_1) \longrightarrow \alpha_{01} = \alpha_1|_{D_+(xy)} - \alpha_0|_{D_+(xy)}$$

$\check{H}^0(\mathcal{U}, \mathcal{O}_X) = H^0(X, \mathcal{O}_X) = k$

$\check{H}^1(\mathcal{U}, \mathcal{O}_X) = 0$

Lemma: With the notation as above, there is a natural map $\mathcal{F} \to \underline{C}^\bullet(\mathcal{U}, \mathcal{F})$ sending $\alpha \in \mathcal{F}(V)$ to $(\alpha|_{U_i \cap V})_{i \in I}$.

This map induces isoms

$$\mathcal{F} = \underline{\check{H}}^0(\mathcal{U}, \mathcal{F}) := H^0(\underline{C}^\bullet(\mathcal{U}, \mathcal{F}))$$

$$\cdot\ \Gamma(T, \mathcal{F}) = \check{H}^0(\mathcal{U}, \mathcal{F}) := H^0(\Gamma(T, \underline{C}^\bullet(\mathcal{U}, \mathcal{F})))$$

Thm: The map $\mathcal{F} \to \underline{C}^\bullet(\mathcal{U}, \mathcal{F})$ produced above gives a resolution of $\mathcal{F}$ by $\underline{C}^\bullet(\mathcal{U}; \mathcal{F})$. That is

① $\underline{\check{H}}^i(\mathcal{U}, \mathcal{F}) = 0 \quad \forall i > 0$, ② $\mathcal{F} \to \underline{\check{H}}^0(\mathcal{U}, \mathcal{F})$ is an isom.

Pf ② is fine, for ① see Hart, Lemma 4.2.

Thm. T, $\mathcal{U}$ be as above. For a flasque sheaf of ab gps $\mathcal{F}$,

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = 0 \quad \forall i > 0.$$

pf. Since $\mathcal{F}$ is flasque, each $\mathcal{F}|_{U_{i_0 \ldots i_p}}$ is flasque

$\Rightarrow (j_{i_0 \ldots i_p})_* (\mathcal{F}|_{U_{i_0 \ldots i_p}})$ is flasque

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$\Rightarrow$ $\underline{C}^p(\mathcal{U}, \mathcal{F})$ being $\prod$ of flasque sheaves is flasque.

Thus $\underline{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a flasque resolution of $\mathcal{F}$.

Thus $\Gamma(T, \underline{C}^\bullet(\mathcal{U}, \mathcal{F})) = C^\bullet(\mathcal{U}, \mathcal{F})$ computes the sheaf cohomologies of $\mathcal{F}$.

Since $\mathcal{F}$ is flasque

$$0 = H^i(T, \mathcal{F}) = H^i(C^\bullet(\mathcal{U}, \mathcal{F})) = \check{H}^i(\mathcal{U}, \mathcal{F}) \quad \forall i > 0.$$

Thm. ① Let X be noetherian, separated scheme. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a finite affine open covering. For a q.coh sheaf $\mathcal{F}$,

$$H^i(X, \mathcal{F}) \simeq \check{H}^i(\mathcal{U}, \mathcal{F})$$

② X, $\mathcal{U}$ as above, $f: X \to Y$ be a separated map

Then $R^i f_* \mathcal{F} \simeq H^i(f_* \underline{C}^\bullet(\mathcal{U}, \mathcal{F}))$

Pf. ① First we show that $\{\check{H}^i(\mathcal{U}, -)\}_{i \in \mathbb{N}}$ is a universal δ-functor from $\mathrm{Qcoh}(X) \to \mathrm{Ab\,gp}$.

- The additivity and functoriality of each $\check{H}^i(\mathcal{U}, -)$ is clear.
- We need to check that given a short exact seq

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \quad \text{in } \mathrm{Qcoh}(X),$$

$\{\check{H}^i(\mathcal{U}, -)\}_{i \in \mathbb{N}}$ produces a long exact seq.

Note (i) Each $U_{i_0 \ldots i_p}$ is affine as X is separated.

q.coh $\Rightarrow$ $0 \to \mathcal{F}'(U_{i_0 \ldots i_p}) \to \mathcal{F}(U_{i_0 \ldots i_p}) \to \mathcal{F}''(U_{i_0 \ldots i_p}) \to 0$ is exact $\forall\, i_0 < i_1 \ldots < i_p$, $\forall p$.

$\Rightarrow$ $0 \to \Gamma(X, \underline{C}^p(\mathcal{U}, \mathcal{F}')) \to \Gamma(X, \underline{C}^p(\mathcal{U}, \mathcal{F})) \to \Gamma(X, \underline{C}^p(\mathcal{U}, \mathcal{F}'')) \to 0$ is exact

$\Rightarrow$ we have a short exact seq of complexes

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}') \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{F}'') \to 0$$

This produces the desired long exact seq of Čech cohomology gps.

Universality of $\{\check{H}^i(\mathcal{U}, -)\}$. It's enough to show

Universality of $\{\check{H}^i(\mathcal{U},-)\}$. It's enough to show

That each $\check{H}^i(\mathcal{U},-)$ is effaceable in $Qcoh(X)$, for each $i>0$

Given $\mathcal{F} \in Qcoh(X)$, choose an embedding

$0 \to \mathcal{F} \to \mathcal{J}$ where $\mathcal{J}$ is q.coh flasque

Since $H^i(\mathcal{U},\mathcal{J}) = 0 \ \forall i>0$, we are done.

On the other hand $\{H^i(X,-)\}_{i\in\mathbb{N}}$ is a universal δ-functor from $Qcoh(X) \to Abgp$

Universality follows since every object can be embedding into a flasque q.coh sheaf and flasque objects are $\Gamma(X,-)$ acyclic.

Since $\check{H}^0(\mathcal{U},\mathcal{F}) \xrightarrow{\sim} H^0(X,\mathcal{F})$ we are done by universality.

② Since f is separated, for any affine open $V \subseteq Y$ and $U \subseteq X$,

$f^{-1}(V) \cap U$ is affine.

Finish the rest as H.W (follows from ①).

End of 13.12.24 Lecture