Sheaf Cohomology

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03.12.25 lecture continued.

Given a short exact seq of sheaves on a scheme

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

The induced map $\Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}'')$ need not be surjective.

Can we systematically measure the failure of this right exactness? Sheaf cohomology theory produces an answer. Indeed, we will attach functors $\{H^i(X,-)\}_{i \in \mathbb{N}} : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X(X)\text{-mod}$ such that the above exact seq will produce an exact seq

$$0 \to \underset{\Gamma(X,\mathcal{F}')}{\underset{\parallel}{H^0(X,\mathcal{F}')}} \to \underset{\Gamma(X,\mathcal{F})}{\underset{\parallel}{H^0(X,\mathcal{F})}} \to \underset{\Gamma(X,\mathcal{F}'')}{\underset{\parallel}{H^0(X,\mathcal{F}'')}}$$

$$\to H^1(X,\mathcal{F}') \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{F}'')$$

$\vdots$

i.e the failure of surjectivity $\left(= \dfrac{H^0(X,\mathcal{F}'')}{\operatorname{Im}(X,\mathcal{F})}\right)$ is the kernel of the map $H^1(X,\mathcal{F}') \to H^1(X,\mathcal{F})$ and so on.

Instead of working with the right exactness of the special functor $\Gamma(X,-)$, we consider the same problem for more general functors and devise an analogous solution.

Convention: Functors are between abelian categories, all functors are additive.

Eg: $\Gamma : \mathrm{Mod}\,\mathcal{O}_X \to \mathrm{Mod}\,\mathcal{O}_X(X)$, for a map of ringed spaces or schemes $f : X \to Y$, $f_* : \mathrm{Mod}\,\mathcal{O}_X \to \mathrm{Mod}\,\mathcal{O}_Y$; $f_* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$, for $\mathcal{F} \in \mathrm{Mod}\,\mathcal{O}_X$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},-)$.

Def.
- Covariant δ-functor (see Hart)
- universal δ-functor.
- uniqueness of universal δ-functor.
- effaceable δ-functors.
- Injective objects in abelian category.
- Notion of having enough injectives
- Injective resolutions.

§ Generalities:

Lemma. Effaceable δ-functors are universal.

Pf. Let $\{T^i\}_{i\in\mathbb{N}}$ be an effaceable δ-functor $\mathcal{A}\to\mathcal{B}$. Given another δ-functor $\{S^i\}_{i\in\mathbb{N}}$ and a natural transformation $f: T^0\to S^0$, construct $f^n: T^n\to S^n$ s.t $f^0=f$ by induction on n.

Set $f^0=f$, Suppose $f^1,\dots,f^n$ are already constructed.

Given $A\in\mathcal{A}$, fix an exact seq

$$0\to A\to I\to I''\to 0 \quad \text{s.t } F^{n+1}(A)\to F^{n+1}(I) \text{ is}$$
the zero map.

Have a diag

$$\begin{tikzcd}
\to T^n(A) \arrow[r]\arrow[d,"f^n(A)"] & T^n(I) \arrow[r]\arrow[d,"f^n_{(I)}"] & T^n(I'') \arrow[r]\arrow[d,"f^n_{(I'')}"] & T^{n+1}(A) \arrow[r,"0"]\arrow[d,dashed] & T^{n+1}(I)\\
\to S^n(A) \arrow[r] & S^n(I) \arrow[r] & S^n(I'') \arrow[r] & S^{n+1}(A) \arrow[r] & S^{n+1}(I)
\end{tikzcd}$$

$$\begin{tikzcd}
T^{n+1}(A) = \operatorname{coker}(T^n(I)\to T^n(I'')) \arrow[d] \arrow[dr, bend right, "f^{n+1}(A)"'] & \\
\operatorname{coker}(S^n(I)\to S^n(I'')) = \operatorname{Ker}(S^{n+1}(A)\to S^{n+1}(I)) & S^{n+1}(A) \arrow[from=l, hook]
\end{tikzcd}$$

We need to check

(i) That $f^{n+1}(A)$ does not depend on the choice of $0\to A\to I$

(ii) functoriality, given $A\to A'$, have

$$\begin{tikzcd}
T^{n+1}(A) \arrow[r,"f^{n+1}(A)"'] \arrow[d] & S^{n+1}(A) \arrow[d]\\
T^{n+1}(A') \arrow[r,"f^{n+1}(A')"'] & S^{n+1}(A')
\end{tikzcd}$$
commutative.

(i) Given $0\to A\to I$, $\quad 0\to A\to I'$
consider $\quad 0\to A\to I\oplus I' \quad$ (diag)

Have
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r]\arrow[d,"id", equal] & I \arrow[r]\arrow[d, shift right]& I/A \arrow[r]\arrow[d, shift right] & 0\\
0 \arrow[r] & A \arrow[r] & I\oplus I' \arrow[r]\arrow[u, shift right] & I\oplus I'/A \arrow[r]\arrow[u, shift right] & 0
\end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal, "\mathrm{id}"] & I \arrow[r] \arrow[d, shift right] & I/A \arrow[r] \arrow[d, shift right] & 0 \\ 0 \arrow[r] & A \arrow[r] & I \oplus I' \arrow[r] \arrow[u, shift right] & I \oplus I'/A \arrow[r] \arrow[u, shift right] & 0 \end{tikzcd}$$

This gives

$$\begin{tikzcd} T^n(A) \arrow[r] \arrow[d, "\mathrm{id}"] & T^n(I) \arrow[r] \arrow[d] & T^n(I/A) \arrow[r] \arrow[d] & T^{n+1}(A) \arrow[d, "\mathrm{id}"] \\ T^n(A) \arrow[r] & T^n(I \oplus I') \arrow[r] & T^n(I \oplus I'/A) \arrow[r] & T^{n+1}(A) \end{tikzcd}$$

showing the extensions given by $0 \to A \to I$ and $0 \to A \to I \oplus I'$ are the same.
Now compare $0 \to A \to I'$ and $0 \to A \to I \oplus I'$.

(iii) Given $0 \to A \xrightarrow{\alpha} I_A$, $0 \to B \xrightarrow{\beta} I_B$, and $g : A \to B$, replace $0 \to A \to I_A$ by

$0 \to A \xrightarrow{(\alpha,\, \beta \cdot g)} I_A \oplus I_B$, Then $T^{n+1}(A) \longrightarrow T^{n+1}(I_A \oplus I_B)$
$= T^{n+1}(I_A) \oplus T^{n+1}(I_B)$
$(T^{n+1}\alpha,\ T^{n+1}\beta \cdot T^{n+1}(g)) = 0$

Hence a diagram

$$(*)\quad \begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "g"] & I_A \oplus I_B \arrow[d, "pr_2"] \\ 0 \arrow[r] & B \arrow[r] & I_B \end{tikzcd}$$

which induces $f^{n+1}(g) : T^{n+1}(A) \longrightarrow T^{n+1}(B)$

The commutativity of the left square below is clear. Since the biggest possible square of the diag below follows from the def of $f^{n+1}(A)$, $f^{n+1}(B)$ and the well definedness. The commutativity of the right square also follows

$$\begin{tikzcd} \operatorname{coker}(T^n(I_A \oplus I_B) \to T^n(A)) = T^{n+1}(A) \arrow[r, "f^{n+1}(A)"] \arrow[d, "\text{coming from } *"'] \arrow[d, phantom, shift left=8, "\circlearrowleft"] & S^{n+1}(A) \arrow[d, "S^{n+1}(g)"] \\ \operatorname{coker}(T^n(I_B) \to T^n(B)) = T^{n+1}(B) \arrow[r, "f^{n+1}(B)"'] & S^{n+1}(B) \end{tikzcd}$$

End of 03.12.25 lecture

Thm. Let $\mathcal{A}$ be an abelian category with enough injectives, $F : \mathcal{A} \to \mathcal{B}$ additive, left exact functor. Then there exists a unique universal δ functor $\{R^i F\}_{i \in \mathbb{N}}$ such that $R^0 F = F$.

$R^i F$ is called the i-th (right) derived functor of F

$R^0F = F$..

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Pf. For $A \in \mathcal{A}$, choose an injective resolution

$$A \to I^\cdot, \quad \text{define} \quad R^iF(A) = H^i(F(I^\cdot))$$

- Since any two injective resolutions are homotopic, different choices of resolutions $I^\cdot$ give isom $\{R^iF(A)\}_{i\in\mathbb{N}}$.

- R^iF's are functors: Given $f: M \to N$ in $\mathcal{A}$, choose injective resolutions $M \to I_M^\cdot$, $N \to I_N^\cdot$. Then f can be extended to a map of complexes: $f^\bullet : I_M^\cdot \to I_N^\cdot$. Thus we get a map $F(I_M^\cdot) \to F(I_N^\cdot)$ and hence a map between cohomologies

$$R^iF(M) \longrightarrow R^iF(N)$$

- δ-functor property: Given a short exact seq in $\mathcal{A}$

$$0 \to M' \to M \to M'' \to 0.$$

by 'Horseshoe lemma' one can choose injective resolutions which fit into short exact seq of complexes

$$0 \to I_{M'}^\cdot \to I_M^\cdot \to I_{M''}^\cdot \to 0,$$

Moreover for each i, $0 \to I_{M'}^i \to I_M^i \to I_{M''}^i \to 0$ is split exact.

So we get, short exact sequence of complexes

$$0 \to F(I_{M'}^\cdot) \to F(I_M^\cdot) \to F(I_{M''}^\cdot) \to 0$$ (The exactness follows from splitting).

This gives the long exact sequence involving $\{R^iF\}_{i\in\mathbb{N}}$.

- $R^0F(A) = A$; F is left exact so $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact $\Rightarrow H^0(F(I^\cdot)) \cong F(A)$.

Prop. If $I \in \mathcal{A}$ is an injective object. Then $R^iF(I) = 0, \forall i > 0$.

Pf. Take the injective resolution $I \to J^\cdot$ where $J^0 = I$, $J^i = 0$ $\forall i > 0$. $\square$.

Rmk. We can play the same game for right exact functors when there are enough projectives.

$i \to j$ where $j = 1, \dots, N$.

Rmk. We can play the same game for right exact functors when there are enough projectives.

Prop: $(X, \mathcal{O}_X)$ be a ringed space. Then the category of $\mathcal{O}_X$-mods have enough injective objects.

Pf. We assume that for a ring A, Mod_A has enough injectives.

For $x \in X$, let $\underline{\{x\}}$ be the ringed space whose underlying top space is $\{x\}$ and the sheaf of ring is the stalk $\mathcal{O}_{X,x}$. Denote the natural map $\underline{\{x\}} \to X$ by j_x.

Given $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, for each $x \in X$, fix an embedding $\mathcal{F}_x \to I_x$ of $\mathcal{O}_{X,x}$-mods, where I_x is injective.

Define $I_{\mathcal{F}} = \prod_{x \in X} I_x$. We argue that $I_{\mathcal{F}}$ is injective $\mathcal{O}_X$-mod.

The $\mathcal{O}_X$-linear str on $I_{\mathcal{F}}$ is clear. It suffices to prove the exactness of $\mathrm{Hom}_{\mathcal{O}_X}(-, I_{\mathcal{F}})$.

Note for $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, I_{\mathcal{F}}) \xrightarrow{\sim} \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$.

Since $\mathcal{G} \to \mathcal{G}_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$ is exact $\forall x$, so is their product. So we are done.

Def/Notation: $(X, \mathcal{O}_X)$ ringed space, $R^i\Gamma(X, -) =: H^i(X, -)$

- For a topological space T, we can consider the category of sheaves of abelian groups and define $H^i(T, -) := R^i\Gamma(-)$

note a sheaf of abelian gps can be thought of as a sheaf of module over the sheaf of rings $\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ is the sheafification of the constant sheaf $U \to \mathbb{Z}$.

- $F = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)$ $\quad R^iF(-) =: \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, -)$.

Def. $F : \mathcal{A} \to \mathcal{B}$ additive left exact. $J \in \mathcal{A}$ is called F-acyclic if $R^iF(J) = 0$ $\forall i > 0$.

<u>Def.</u> $F: \mathcal{A} \to \mathcal{B}$ additive left exact. $J \in \mathcal{A}$ is called F acyclic if $R^iF(J) = 0 \;\forall i > 0$.

<u>Prop.</u> $F: \mathcal{A} \to \mathcal{B}$ (additive) left exact functor. For $A \in \mathcal{A}$, suppose there is a resolution of A by F acyclic objects, i.e. $\exists$ complex

$J^0 \to J^1 \to \cdots \in \mathcal{A}$ such that each J^i is acyclic and $H^0(J^\cdot) \xrightarrow{\sim} A$ and $H^i(J^\cdot) = 0 \;\forall i > 0$.

Then $R^iF(A) \xrightarrow{\sim} H^i(F(J^\cdot))$

<u>Pf:</u> Given an injective resolution $A \to I^\cdot$, we have a map of complexes

$$\begin{array}{ccccccccc} 0 & \to & A & \to & I^0 & \to & I^1 & \to & \cdots \\ & & {\scriptstyle id}\| & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & J^0 & \to & J^1 & \to & \cdots \end{array}$$

This induces a map $R^iF(A) = H^i(F(I^\cdot)) \to H^i(F(J^\cdot))$

This map is an isom $\forall i$, but we don't verify that. Instead we show $R^iF(A) \xrightarrow{\sim} H^i(F(J^\cdot))$ abstractly.

- For $i = 0$, $R^0F(A) = A \xrightarrow{\sim} H^0(F(J^\cdot)) \xrightarrow{\sim} A$ as F is left exact.
- We induce on i. Suppose we have isom $\forall A \in \mathcal{A}$, and $i \leq n$

Let $A' = \operatorname{coker}(A \to J^0)$, then $J^\cdot[1] = 0 \to J^1 \to J^2 \to \cdots$ a resolution $0 \to A' \to J^1 \to J^2 \to \cdots$

The exact seq $0 \to A \to J^0 \to A' \to 0$ gives

$$\begin{array}{ccccccccccccc} 0 \to F(A) \to & F(J^0) & \to & F(A') & \to & R^1F(A) & \to 0 \to R^1F(A') \to R^2F(A) \to 0 \\ & \| & & \| & & \downarrow\wr & \\ & F(J^0) & \to & \ker(F(J^1) & \to & H^1(F(J^\cdot)) & \\ & & & \downarrow & & & \\ & & & F(J^2)) & & & \end{array}$$

$\Rightarrow \quad R^1F(A) \xrightarrow{\sim} H^1(F(J^\cdot))$

and $\quad R^iF(A') \xrightarrow{\sim} R^{i+1}F(A) \quad \forall i \geq 1$

Since $\quad \begin{array}{ccc} R^iF(A') & \xrightarrow{\sim} & H^i(J^\cdot[1]) \quad \text{for } i \leq n \\ \downarrow\wr & & \downarrow\wr \\ R^{i+1}F(A) & \xrightarrow{\sim} & H^{i+1}(J^\cdot) \end{array}$

$$R^{i+1}F(A) \xrightarrow{\sim} H^{i+1}(J^\bullet)$$

we are done. ∎

<u>Def:</u> A sheaf $\mathcal{F}$ on a topological space is called flasque if for any two open's in X U, V s.t $U \subseteq V$, $\mathcal{F}(V) \to \mathcal{F}(U)$ is sur.

<u>Prop:</u> $(X, \mathcal{O}_X)$ ringed space. Every injective $\mathcal{O}_X$-mod is flasque

<u>Pf</u> For every open $U \subseteq X$, define $(j_U)_! \mathcal{F}$

to be the sheafification of

$$V \longmapsto 0 \quad \text{if } V \nsubseteq U$$
$$V \longmapsto \mathcal{F}(V) \quad \text{if } V \subseteq U$$

Given $V \subseteq U$ opens

Have an injection $0 \to (j_V)_! \mathcal{O}_X \to (j_U)_! \mathcal{O}_X$

Since $\mathcal{I}$ is injective, Hom $(-, \mathcal{I})$ gives a surjection

$$\begin{array}{ccccc} 0 & \leftarrow & \mathrm{Hom}_{\mathcal{O}_X}((j_V)_! \mathcal{O}_X, \ ) & \leftarrow & \mathrm{Hom}_{\mathcal{O}_X}((j_U)_! \ , \ ) \\ & & \| & & \| \\ 0 & \leftarrow & \mathcal{I}(V) & \longleftarrow & \mathcal{I}(U) \end{array}$$

Start of 10.12.25 lecture

End of 5.12.25 lecture.

<u>Prop:</u> $(X, \mathcal{O}_X)$ ringed space. For an exact ax $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $\mathrm{Mod}_{\mathcal{O}_X}$

if $\mathcal{F}'$ is Flasque.

(i) Then $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$ is exact

(ii) For a flasque sheaf of abelian groups $\mathcal{G}$, $H^i(X, \mathcal{G}) = 0 \quad \forall i > 0$.

<u>Pf.</u> Given $s \in \Gamma(X, \mathcal{F}'')$, by Zorn's lemma, choose a maximal elt in the non-empty set

$$\{ (U, t) \mid \phi \neq U \underset{\text{open}}{\subseteq} X, \ t \in \mathcal{F}(U) \text{ s.t } t \text{ maps to } s|_U \},$$

call it (U_s, t_s). If $U_s \neq X$, pick $x \in X - U_s$

$\exists$ a open nbhd V of x and $t_V \in \mathcal{F}(V)$ s.t t_V maps to $s_{|V}$.

$$t_V|_{U_s \cap V} - t_s|_{U_s\cap V} \in \mathcal{F}'(U_s \cap V).$$

Since $\mathcal{F}'$ is flasque, $\exists$ $\tilde{t} \in \mathcal{F}'(X)$ mapping to the difference above. So $t_V - \tilde{t}_{|V}$ and t_s agree on $U_s \cap V$. Then $\exists!$ $t' \in \mathcal{F}(V \cup U_s)$ s.t $t'|_{U_s} = t_s$. So $(V \cup U_s, t')$ is bigger than (U_s, t_s) contradicting the maximality.

Then $U_s = X$.

(ii) Choose an injection $\mathcal{G} \to \mathcal{I}$ where $\mathcal{I}$ is an injective $\mathcal{O}_X$-mod.

Show that $\mathcal{I}/\mathcal{G}$ is flasque by considering the diag and that $\mathcal{I}, \mathcal{G}|_U, \mathcal{I}|_U$ are flasque

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{G}(X) & \to & \mathcal{I}(X) & \to & \mathcal{I}/\mathcal{G}(X) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{G}(U) & \to & \mathcal{I}(U) & \to & \mathcal{I}/\mathcal{G}(U) & \to & 0 \end{array}$$

Long exact sequence of sheaf cohomology implies

$$0 \to H^0(X,\mathcal{G}) \to H^0(X;\mathcal{I}) \to H^0(X, \mathcal{I}/\mathcal{G}) \to H^1(X,\mathcal{G}) \to \overset{0}{\overset{\shortparallel}{H^1(X,\mathcal{I})}} \to H^1(X, \mathcal{I}/\mathcal{G}) \to H^2(X,\mathcal{G}) \to 0$$

Since $H^1(X,\mathcal{I}) = 0$, (i) $\Rightarrow$ $H^1(X,\mathcal{G}) = 0$

Moreover we have $H^{i+1}(X,\mathcal{G}) \xrightarrow{\sim} H^i(X, \mathcal{I}/\mathcal{G})$ $\forall i > 0$

By induction show that $H^i(X,\mathcal{G}) = 0$ $\forall$ $i>0$.

<u>Proof:</u> $(X, \mathcal{O}_X)$ ringed space, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$.

(i) Given a flasque resolution $\mathcal{F} \to \mathcal{J}^\bullet$ i.e each $\mathcal{J}^i$ is flasque, $\mathcal{F} \xrightarrow{\sim} H^0(\mathcal{J}^\bullet)$, $H^i(\mathcal{J}^\bullet) = 0$ $\forall$ $i > 0$

The i-th cohomology of $\Gamma(X, \mathcal{J}^\bullet)$ computes $H^i(X,\mathcal{F})$

(ii) T.F.D.C $\forall$ $i \geq 0$

$$\begin{array}{ccc} \text{sheaves of abelian groups} = \mathrm{Mod}_{\underline{\mathbb{Z}}} & \xrightarrow{H^i(X,-)} & \\ \uparrow & & \\ \mathrm{Mod}_{\mathcal{O}_X} & \xrightarrow{\quad} & \text{abelian groups} \end{array}$$

abelian groups

$$\begin{array}{ccc} \uparrow & = & \longrightarrow \\ \mathrm{Mod}_{\mathcal{O}_X} & \xrightarrow[H^i(X,-)]{} & \text{abelian groups} \end{array}$$

i.e for an $\mathcal{O}_X$-mod $\mathcal{F}$, the sheaf cohomology of $\mathcal{F}$ obtained by resolving by injective sheaves of abelian gps and injective sheaves of $\mathcal{O}_X$-mods are the same.

Rmk. Let $(X, \mathcal{O}_X)$ be a ringed space, $F \in \mathrm{Mod}_{\mathcal{O}_X}$. One can explicitly write flasque resolutions of F. For $x \in X$, $\{x\}$ be the ringed space whose underlying top space is x and the sheaf of rings is $\mathcal{O}_{X,x}$. Let $j_x : \{x\} \to X$ be the canonical map. Take

$$J_F = \prod_{x \in X} (j_x)_* F_x$$

check that J^0 is flasque sheaf of $\mathcal{O}_X$-modules. Note that the canonical map $F \to J_F$ is injective. Then $J^0 = J_F$. Extend this to ~~a~~ flasque resolution by taking $J^1 = J_{J_F/F}$, $J^2 = J_{\mathrm{coker}(J^0 \to J^1)}$ and so on.

Thm: ① A noeth ring, I be an injective mod. Then $\tilde{I}$ is a flasque sheaf.

② On a noeth scheme any q-coh $\mathcal{O}_X$-mod can be resolved by a complex of q.coh flasque $\mathcal{O}_X$-mods.

Pf. Step 1: For an ideal $J \subseteq A$. Set

$$\Gamma_J(I) = \{ x \in I \mid J^n . x = 0 \text{ for some } n \in \mathbb{N} \}$$

$$= \{ x \in I \mid \tfrac{x}{1} \in I_p \text{ is zero if } p \notin V(J) \}.$$

So $\Gamma_J(I)$ only depends on $V(J)$ and I.

So $\Gamma_J(I) = \Gamma_{V(J)}(I)$

Claim: $\Gamma_J(I)$ is an injective A-mod. (Lemma 3.2, Hart)

Step 2: For any $f \in A$, the map $I \to I_f$ is surjective. (Lemma 3.3, Hart)

Noeth induction: T noeth top space (i.e every decreasing chain of closed sets stabilize). Let P be a property of closed subset such that for every closed $Z \subseteq T$ if P holds for every proper closed subset of Z, P holds for T. Then P holds for X. [Note by our assumption P holds for $\emptyset$]

Pf of noeth induction:

Consider the collection of closed subsets on which P fails. If this set is non-empty there must be a smallest elt. But our assumption contradicts that. $\square$

X scheme, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$

Define $\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$

X scheme, $\mathcal{F} \in$ Modules.

Define $\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$

For $s \in \mathcal{F}(U)$, $\mathrm{supp}(s) = \{x \in U \mid s_x \neq 0\}$

$X = \mathrm{Spec}(A)$. We say that a closed subset Z has property P if for every injective A-module M with $\mathrm{Supp}(M) \subseteq Z$ $\tilde{M}$ is flasque.

Let Z be a closed subset s.t every proper closed subset has property P. Take an injective mod M s.t $\mathrm{Supp}\, \tilde{M} \subseteq Z$.

Given $U \underset{\text{open}}{\subseteq} X$, if $U \cap Z = \emptyset$,

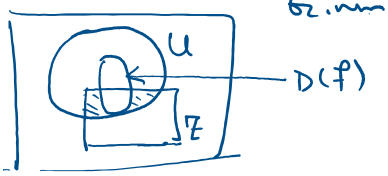
$\tilde{M}(X) \longrightarrow \tilde{M}(U) = 0$ is surjective.

Assume $U \cap Z \neq \emptyset$ choose $f \in A$ s.t $D(f) \subseteq U$ and $D(f) \cap Z \neq \emptyset$. $Z' = X - D(f) = V(f)$

Define $M' = \{x \in M \mid f^n . x = 0$ for some n depending on $x\}$

Note $\tilde{M}'(U) = \{s \in \tilde{M}(U) \mid \mathrm{supp}(s) \subseteq U \cap Z'\}$

(+) Since $\mathrm{supp}(M') \subseteq V(f) \cap \mathrm{supp}(M) \subseteq V(f) \cap Z \subsetneq Z$, $\tilde{M}'$ is flasque by our induction hypothesis



Given $s \in \Gamma(U, \tilde{M})$, $\exists$ $s' \in \Gamma(X, \tilde{M})$ s.t $s'|_{D(f)} = s|_{D(f)}$

Then $s - s'|_U \in \Gamma(U, \tilde{M})$ has support in $V(f) \cap Z$

Then $s - s'|_U \in \Gamma(U, \tilde{M}') \subseteq \Gamma(U, \tilde{M})$. Since $\tilde{M}'$ is flasque

[By our induction hypo, since $\mathrm{supp}\ (\Gamma_{Z'}(M)) \subsetneq Z$, $\widetilde{\Gamma_{Z'}(M)}$ is flasque] Choose $t \in \tilde{M}'(X) = M' \subseteq M$ s.t $t|_U = s - s'|_U$

Then $(s' + t)|_U = s$, $s' + t \in \Gamma(X, \tilde{M}) = M$.

Then $(s'+t)|_U = s$, $s'+t \in \Gamma(X, \tilde{M}) = M$.

Prop: X be a noetherian scheme. Every q.coh $\mathcal{O}_X$ mod $\mathcal{F}$ admits a resolution by <u>q.coh</u> flasque sheaves.

Pf. Given $\mathcal{F} \in \mathrm{Qcoh}(X)$, enough to produce an injection $0 \to \mathcal{F} \to \mathcal{I}$, where $\mathcal{I}$ is a <u>q.coh</u> flasque $\mathcal{O}_X$-mod.

Choose an affine open covering $X = \bigcup_{i=1}^{n} U_i$.

For each i, choose an injection $0 \to \mathcal{F}|_{U_i} \to I_i$ where I_i is an injective $\mathcal{O}_X(U_i)$-mod. Denote the immersion $U_j \to X$ by i_j.

Then get an injection

$$0 \to \mathcal{F} \to \bigoplus_{j=1}^{n} (i_j)_*(\mathcal{F}|_{U_j}) \to \bigoplus_{j=1}^{n} (i_j)_*(\tilde{I}_j)$$

$$f \in \mathcal{F}(V) \longmapsto (f|_{V \cap U_j})$$

Claim. $\bigoplus_{j=1}^{n} (i_j)_*(\tilde{I}_j)$ is flasque.

Pf. $\tilde{I}_j$ is flasque on U_j, so $(i_j)_*(\tilde{I}_j)$ is flasque, so is the direct sum.

Thm. Let X be a noeth affine scheme. $\mathcal{F} \in \mathrm{Qcoh}(X)$.

$$H^i(X, \mathcal{F}) = 0 \quad \forall \quad i > 0.$$

Rmk. THE ABOVE THM IS TRUE WITHOUT ANY NOETH HYPOTHESIS.

Pf. Let $X = \mathrm{Spec}(A)$, $\mathcal{F} \cong \tilde{M}$ for some $M \in \mathrm{Mod}_A$.

Choose an injective resolution $0 \to M \to I^0 \to I^1 \to$ in Mod_A.

Then have a flasque resolution in $\mathrm{Mod}_{\mathcal{O}_X}$

$$0 \to \tilde{M} \to \tilde{I}^0 \to \tilde{I}^1 \to \cdots$$

Since flasque sheaves are $\Gamma(X, -)$ acyclic

$$H^i(X, \tilde{M}) = H^i(\Gamma(\tilde{I}^\bullet))$$

$$= H^i(I^0 \to I^1 \to \cdots \to I^i \to I^{i+1} \to \cdots)$$

$= H^i(I^0 \to I^1 \to \cdots \to I^i \to I^{i+1} \to \cdots)$

$= 0$ for $i > 0$. $[\because I^j$ q.coh $\tilde{I}^j(X) \simeq I^j$

<u>Thm:</u> Let X be a noeTh scheme. T.F.A.E

① X is affine

② $\forall$ q.coh $\mathcal{O}_X$-mod $\mathcal{F}$, $H^i(X, \mathcal{F}) = 0$ $\forall i > 0$.

③ For all q.coh ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$. $H^i(X, \mathcal{J}) = 0$ $\forall i > 0$.

<u>Pf.</u> It suffices to prove (2) $\Rightarrow$ (1). For This we grant ourselves The following criterion of affineness.

<u>Fact.</u> (Ex 2.17, Hart) A scheme Y is affine iff $\exists$ $f_1, \ldots, f_r \in \Gamma(Y, \mathcal{O}_Y)$ such That each open subset Y_{f_i} is affine and The ideal $\langle f_1, \ldots, f_r \rangle = \Gamma(Y, \mathcal{O}_Y)$.

For $x \in X$, we will find $f_x \in \Gamma(X, \mathcal{O}_X)$ such That X_{f_x} is an affine nbh of x. To This end, choose an affine open nbhd U of x and a q.coh $\mathcal{J} \subseteq \mathcal{O}_X$ such That $V(\mathcal{J}) = X - U$.

Let $i : \operatorname{Spec}(k(x)) \longrightarrow X$ be The canonical map.

We have an exact seq $0 \to \mathcal{J} \cup \{x\} \to \mathcal{J} \to i_*(k(x)) \to 0$

Since $H^1(X, \mathcal{J} \cup \{x\}) = 0$, we have a surjection

$H^0(X, \mathcal{J}) \longrightarrow H^0(X, i_*(k(x))) = k(x)$

Take an $f_x \in H^0(X, \mathcal{J})$ which maps to $1 \in k(x)$. Note $f_x|_U \in \Gamma(U, \mathcal{O}_X)$ and $X_{f_x} = U_{f_x}$ is affine and contains x.

Since X is noeTherian, we have a finite covering

$$X = \bigcup_{i=1}^n X_{f_{x_i}}.$$

The rest of The proof follows from The claim:

<u>claim:</u> $\bigoplus_{i=1}^n H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X)$ is surjective.

$1 \longmapsto f_{x_i}$

<u>Pf.</u> Consider The exact sequence

$$0 \to \mathcal{F} \to \bigoplus \mathcal{O}_X \to \mathcal{O}_X \to 0$$

$1 \longmapsto f_{x_i}$

Now, we prove $H^1(X, \mathcal{F}) = 0$: The claim will follow from long exact sequence of cohomology.

Consider The filtration on $\mathcal{F}$ given by $\{\mathcal{F} \cap \bigoplus^m \mathcal{O}_X\}_{m=0}^n$ in The increasing filtration we add one copy of $\mathcal{O}_X$ at a time.

. . . q.coh. so is $\mathcal{F}_{m+1}/\mathcal{F}_m$ for every

Consider the increasing filtration we add one copy of $\mathcal{O}_X$ at a time. The $\mathcal{F}$ and hence each $\mathcal{F}_m$ is q. coh; so is $\mathcal{F}_{m+1}/\mathcal{F}_m$ for every $0 \le m \le n-1$. Now note that $\mathcal{F}_{m+1}/\mathcal{F}_m$ is a subsheaf of $\mathcal{O}_X$ so is an ideal sheaf. We prove by inducting on m that $H^1(X, \mathcal{F}_m) = 0$ $\forall m$.

To this end $m=1$ case is clear.

For the induction step use the exact seq

$$0 \to \mathcal{F}_m \to \mathcal{F}_{m+1} \to \mathcal{F}_{m+1}/\mathcal{F}_m \to 0$$

and long exact seq of cohomology (note $H^1(X, \mathcal{F}_{m+1}/\mathcal{F}_m) = 0$ $\forall m$).

Thm: X be a noetherian topological space.

Define $\dim(X) = \sup\{n \mid \exists$ a strict chain of irr closed sets $X_0 \subsetneq X_1 \subsetneq \dots \subseteq X_n\}$

For any sheaf of abelian groups $\mathcal{F}$ on X, $H^i(X, \mathcal{F}) = 0$ $\forall i > \dim(X)$.

We do not prove it in class but prove a weaker version.

For a proof of the above, see Thm 2.7, Ch III Hart.