

Sheaf Cohomology

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03.12.25 lecture continued.

Given a short exact seq of sheaves on a scheme

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

The induced map $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}'')$ need not be surjective.

Can we systematically measure the failure of this right exactness?

Sheaf cohomology theory provides an answer. Indeed, we will attach functors $\{H^i(X, -)\}_{i \in \mathbb{N}}: \mathcal{O}_X\text{-mod} \rightarrow \mathcal{O}_X(X)\text{-mod}$ such that the above exact seq will produce an exact seq

$$0 \rightarrow H^0(X, \mathcal{F}') \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{F}'') \rightarrow H^1(X, \mathcal{F}') \rightarrow H^1(X, \mathcal{F}) \rightarrow H^1(X, \mathcal{F}'') \rightarrow \dots$$

i.e. the failure of surjectivity ($= \frac{H^0(X, \mathcal{F}'')}{\text{Im}(H^0(X, \mathcal{F}))}$) is the kernel of the map $H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{F}'')$ and so on.

Instead of working with the right exactness of the special functor $\Gamma(X, -)$, we consider the same problem for more general functors and devise an analogous solution.

Convention: Functors are between abelian categories, all functors are additive.

Eg: $\Gamma: \text{Mod}_{\mathcal{O}_X} \rightarrow \text{Mod}_{\mathcal{O}_X(X)}$, For a map of ringed spaces on schemes $f: X \rightarrow Y$, $f_*: \text{Mod}_{\mathcal{O}_X} \rightarrow \text{Mod}_{\mathcal{O}_Y}$; $f_x: \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$, For $\mathcal{F} \in \text{Mod}_{\mathcal{O}_X}$, $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, -) =$

- Def. • Covariant \mathcal{S} -functor (see Hart)
- universal \mathcal{S} -functor.
 - uniqueness of universal \mathcal{S} -functor.
 - effable \mathcal{S} -functor.
 - Injective objects in abelian category.
 - Notion of having enough injectives
 - Injective resolutions.

Generalities:

Lemma: Effaceable δ -functors are universal.

Pl. Let $\{T^i\}_{i \in \mathbb{N}}$ be an effaceable δ -functor $A \rightarrow B$.
 Given another δ -functor $\{S^i\}_{i \in \mathbb{N}}$ and a natural transformation $f: T^0 \rightarrow S^0$, construct $f^n: T^n \rightarrow S^n$ s.t. $f^0 = f$ by induction on n .

Set $f^0 = f$, Suppose f^1, \dots, f^n are already constructed.

Given $A \in \mathcal{A}$, fix an exact seq

$$0 \rightarrow A \rightarrow I \rightarrow I'' \rightarrow 0 \text{ s.t. } F^{n+1}(A) \rightarrow F^{n+1}(I) \text{ is}$$

The zero map.

Have a diag

$$\begin{array}{ccccccc} \rightarrow & T^n(A) & \rightarrow & T^n(I) & \rightarrow & T^n(I'') & \rightarrow & T^{n+1}(A) & \xrightarrow{0} & T^{n+1}(I) \\ & \downarrow f^n(A) & & \downarrow f^n(I) & & \downarrow f^n(I'') & & \downarrow & & \\ \rightarrow & S^n(A) & \rightarrow & S^n(I) & \rightarrow & S^n(I'') & \rightarrow & S^{n+1}(A) & \rightarrow & S^{n+1}(I) \end{array}$$

$$T^{n+1}(A) = \text{coker}(T^n(I) \rightarrow T^n(I''))$$

$$\begin{array}{ccc} & \downarrow & \\ & \text{coker}(S^n(I) \rightarrow S^n(I'')) = \text{Ker}(S^{n+1}(A) \rightarrow S^{n+1}(I)) & \\ f^{n+1}(A) \cong & \searrow & \downarrow \\ & & S^{n+1}(A) \end{array}$$

We need to check

(i) That $f^{n+1}(A)$ does not depend on the choice of $0 \rightarrow A \rightarrow I$

(ii) Functoriality, given $A \rightarrow A'$,
 have

$$\begin{array}{ccc} T^{n+1}(A) & \xrightarrow{f^{n+1}(A)} & S^{n+1}(A) \\ \downarrow & & \downarrow \\ T^{n+1}(A') & \xrightarrow{f^{n+1}(A')} & S^{n+1}(A') \end{array}$$

commutative.

(i) Given $0 \rightarrow A \rightarrow I, \dots, 0 \rightarrow A \rightarrow I'$

(i) Given $0 \rightarrow A \rightarrow I, \quad 0 \rightarrow A \rightarrow I'$
 consider $0 \rightarrow A \rightarrow I \oplus I'$ (diag)

Have $0 \rightarrow A \rightarrow I \rightarrow I/A \rightarrow 0$
 $\quad \quad \quad \downarrow \uparrow \quad \quad \downarrow \uparrow$
 $0 \rightarrow A \rightarrow I \oplus I' \rightarrow I \oplus I'/A \rightarrow 0$

This gives $T^n(A) \rightarrow T^n(I) \rightarrow T^n(I/A) \rightarrow T^{n+1}(A)$
 $\quad \quad \quad \downarrow \text{id} \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \text{id}$
 $T^n(A) \rightarrow T^n(I \oplus I') \rightarrow T^n(I \oplus I'/A) \rightarrow T^{n+1}(A)$

showing The extensions given by $0 \rightarrow A \rightarrow I$ and $0 \rightarrow A \rightarrow I \oplus I'$ are the same.

Now compare $0 \rightarrow A \rightarrow I'$ and $0 \rightarrow A \rightarrow I \oplus I'$

(ii) Given $0 \rightarrow A \xrightarrow{\alpha} I_A, \quad 0 \rightarrow B \xrightarrow{\beta} I_B$
 and $g: A \rightarrow B$, replace $0 \rightarrow A \rightarrow I_A$ by

$0 \rightarrow A \rightarrow I_A \oplus I_B$, Then $T^{n+1}(A) \rightarrow T^{n+1}(I_A \oplus I_B)$
 $(\alpha, \beta \cdot g)$ $= T^{n+1}(I_A) \oplus T^{n+1}(I_B)$
 $(T^{n+1} \alpha, T^{n+1} \beta \cdot T^{n+1}(g)) = 0$

Have a diagram

$\begin{array}{ccc} 0 \rightarrow A \rightarrow I_A \oplus I_B & \text{which induces } f^{n+1}(g) & \\ \downarrow g & \downarrow \text{pr}_2 & \downarrow T^{n+1}(g) \\ 0 \rightarrow B \rightarrow I_B & & T^{n+1}(A) \rightarrow T^{n+1}(B) \end{array}$

The commutativity

of the left square below is clear. Since the biggest possible square of the diag below follows from the def of $f^{n+1}(A), f^{n+1}(B)$ and the well definedness. The commutativity of the right square also follows

$\text{coker}(T^n(I_A \oplus I_B) \rightarrow T^n(A)) = T^{n+1}(A) \xrightarrow{f^{n+1}(A)} \mathcal{S}^{n+1}(A)$
 $\quad \quad \quad \downarrow \text{coming from } \alpha \quad \quad \downarrow T^{n+1}(g) \quad \quad \downarrow \mathcal{S}^{n+1}(g)$
 $\text{coker}(T^n(I_B) \rightarrow T^n(B)) = T^{n+1}(B) \xrightarrow{f^{n+1}(B)} \mathcal{S}^{n+1}(B)$

End of 03.12.25 lecture

Thm. Let \mathcal{A} be an abelian category with enough injectives, $F: \mathcal{A} \rightarrow \mathcal{B}$ additive, left exact, functor. Then there exists a unique universal δ functor $\{R^i F\}_{i \in \mathbb{N}}$ such that $R^0 F = F$.

$R^i F$ is called the i -th (right) derived functor of F

Pf. For $A \in \mathcal{A}$, choose an injective resolution

$$A \rightarrow I^\bullet, \text{ define } R^i F(A) = H^i(F(I^\bullet))$$

- Since any two injective resolutions are homotopic, different choices of resolutions I^\bullet give isom $\{R^i F(A)\}_{i \in \mathbb{N}}$.
- $R^i F$'s are functors: Given $f: M \rightarrow N$ in \mathcal{A} , choose injective resolutions $M \rightarrow I_M^\bullet, N \rightarrow I_N^\bullet$. Then f can be extended to a map of complexes: $f^\bullet: I_M^\bullet \rightarrow I_N^\bullet$. Then we get a map $F(I_M^\bullet) \rightarrow F(I_N^\bullet)$ and hence a map between cohomologies

$$R^i F(M) \rightarrow R^i F(N)$$

- δ -functor property: Given a short exact seq in \mathcal{A}

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0.$$

by 'Horseshoe lemma' one can choose injective resolutions which fit into short exact seq of complexes

$$0 \rightarrow I_{M'}^\bullet \rightarrow I_M^\bullet \rightarrow I_{M''}^\bullet \rightarrow 0.$$

Moreover for each i , $0 \rightarrow I_{M'}^i \rightarrow I_M^i \rightarrow I_{M''}^i \rightarrow 0$ is split exact.

So we get, short exact sequence of complexes

$$0 \rightarrow F(I_{M'}^\bullet) \rightarrow F(I_M^\bullet) \rightarrow F(I_{M''}^\bullet) \rightarrow 0 \text{ (The exactness follows from splitting).}$$

This gives the long exact sequence involving $\{R^i F\}_{i \in \mathbb{N}}$.

- $R^0 F(A) = A$; F is left exact so
 $0 \rightarrow F(A) \rightarrow F(I^0) \rightarrow F(I^1) \rightarrow \dots$ is exact
 $\Rightarrow H^0(F(I^*)) \cong F(A)$.

Prop. If I of A is an injective object. Then $R^i F(I) = 0, \forall i > 0$.

Pf. Take the injective resolution
 $I \rightarrow J^0 \rightarrow J^1 \rightarrow \dots$ where $J^0 = I, J^i = 0 \forall i > 0$. \square

Rem. We can play the same game for right exact functors when there are enough projectives.

Proof: (X, \mathcal{O}_X) be a ringed space. Then the category of \mathcal{O}_X -mods have enough injective objects.

Pf. We assume that for a ring A , Mod_A has enough injectives.

For $x \in X$, let $\underline{\{x\}}$ be the ringed space whose underlying top space is $\{x\}$ and the sheaf of rings is the stalk $\mathcal{O}_{X,x}$.
 Denote the natural map $\underline{\{x\}} \rightarrow X$ by ∂_x .

Given $F \in \text{Mod}_{\mathcal{O}_X}$, for each $x \in X$, fix an embedding $F_x \rightarrow I_x$ of $\mathcal{O}_{X,x}$ -mods, where I_x is injective.

Define $I_F = \prod_{x \in X} I_x$. We argue that I_F is injective \mathcal{O}_X -mod.

The \mathcal{O}_X -linear str on I_F is clear. It suffices to prove the exactness of $\text{Hom}_{\mathcal{O}_X}(_, I_F)$.

Note for $G \in \text{Mod}_{\mathcal{O}_X}$, $\text{Hom}_{\mathcal{O}_X}(G, I_F) \cong \prod_{x \in X} \text{Hom}_{\mathcal{O}_{X,x}}(G_x, I_x)$.

Since $G \rightarrow G_x \rightarrow \text{Hom}_{\mathcal{O}_{X,x}}(G_x, I_x)$ is exact $\forall x$, so is their product. So we are done.

Def/ Notation: (X, \mathcal{O}_X) ringed space, $R^i \Gamma(X, _) =: H^i(X, _)$

- for a topological space T , we can consider the category of sheaves of abelian groups and define $H^i(T, _)$
 $:= R^i \Gamma(_)$

of sheaves of abelian groups and define $H^i(\mathcal{T}, -) := R^i\Gamma(-)$

note a sheaf of abelian groups can be thought of as a sheaf of modules over the sheaf of rings $\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ is the sheafification of the constant sheaf $\mathbb{Z} \rightarrow \mathbb{Z}$.

• $F = \text{Hom}_{\mathcal{O}_x}(\mathcal{F}, -)$ $R^i F(-) := \text{Ext}_{\mathcal{O}_x}^i(\mathcal{F}, -)$.

Def. $F: \mathcal{A} \rightarrow \mathcal{B}$ additive left exact. $\mathcal{J} \in \mathcal{A}$ is called F acyclic if $R^i F(\mathcal{J}) = 0 \forall i > 0$.

Prop. $F: \mathcal{A} \rightarrow \mathcal{B}$ (additive) left exact functor.
For $\mathcal{A} \in \mathcal{A}$, suppose there is a resolution of \mathcal{A} by F acyclic objects, i.e. \mathcal{J} complex

$$\mathcal{J}^0 \rightarrow \mathcal{J}^1 \rightarrow \dots \in \mathcal{A} \text{ such that each } \mathcal{J}^i \text{ is acyclic and } H^0(\mathcal{J}^\bullet) \cong \mathcal{A} \text{ and } H^i(\mathcal{J}^\bullet) = 0 \forall i > 0.$$

Then $R^i F(\mathcal{A}) \cong H^i(F(\mathcal{J}^\bullet))$

Pf: Given an injective resolution $A \rightarrow I^\bullet$, we have a map of complexes

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & I^0 & \rightarrow & I^1 & \rightarrow & \dots \\ & & \text{id} \parallel & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & A & \rightarrow & J^0 & \rightarrow & J^1 & \rightarrow & \dots \end{array}$$

This induces a map $R^i F(A) = H^i(F(I^\bullet)) \rightarrow H^i(F(J^\bullet))$

This map is an isom $\forall i$, but we don't verify that. Instead we show $R^i F(A) \cong H^i(F(J^\bullet))$ abstractly.

- for $i=0$, $R^0 F(A) = A \xrightarrow{\cong} H^0(F(J^\bullet)) \xrightarrow{\cong} A$ as F is left exact.
- We induct on i . Suppose we have isom $\forall \mathcal{A} \in \mathcal{A}$, and $i \leq n$

Let $A' = \text{coker}(A \rightarrow J^0)$, then J^\bullet gives a resolution $0 \rightarrow A' \rightarrow J^1 \rightarrow J^2 \rightarrow \dots$

The exact seq $0 \rightarrow A \rightarrow J^0 \rightarrow A' \rightarrow 0$

The exact seq $0 \rightarrow A \rightarrow J^0 \rightarrow A' \rightarrow 0$
gives

$$\begin{array}{ccccccccccc}
 0 & \rightarrow & F(A) & \rightarrow & F(J^0) & \rightarrow & F(A') & \rightarrow & R^1 F(A) & \rightarrow & 0 & \rightarrow & R^1 F(A') & \rightarrow & R^2 F(A) & \rightarrow & 0 \\
 & & & & \parallel & & \parallel & & \downarrow \mathcal{L} & & & & & & & & \\
 & & & & F(J^0) & \rightarrow & \text{Ker}(F(J^1)) & \rightarrow & H^1(F(J^0)) & & & & & & & & \\
 & & & & & & \downarrow & & F(J^2) & & & & & & & &
 \end{array}$$

$$\Rightarrow R^1 F(A) \cong H^2(F(J^0))$$

$$\text{and } R^i F(A') \cong R^{i+1} F(A) \quad \forall i \geq 1$$

$$\begin{array}{ccc}
 \text{Since } R^i F(A') \cong H^i(J^0[2]) & \text{for } i \leq n \\
 \downarrow & \downarrow \\
 R^{i+1} F(A) \cong H^{i+1}(J^0) &
 \end{array}$$

we are done. \square

Def: A sheaf \mathcal{F} on a topological space is called flasque if for any two opens in X U, V s.t $U \subseteq V$, $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$ is sur.

Prop: (X, \mathcal{O}_X) ringed space. Every injective \mathcal{O}_X -mod is flasque

Pf For every open $U \subseteq X$, define $(j_U)_* (\mathcal{F})$

$$\begin{array}{l}
 \text{to be the sheafification of} \\
 v \longmapsto 0 \quad \text{if } v \not\subseteq U \\
 v \longmapsto \mathcal{F}(v) \quad \text{if } v \subseteq U
 \end{array}$$

Given $v \subseteq u$ opens

$$\text{Have an injection } 0 \rightarrow (j_v)_* \mathcal{O}_X \rightarrow (j_u)_* \mathcal{O}_X$$

Since \mathcal{F} is injective, $\text{Hom}(-, \mathcal{F})$ gives a surjection

$$\begin{array}{ccc}
 0 & \leftarrow & \text{Hom}_{\mathcal{O}_X}((j_v)_* \mathcal{O}_X, \mathcal{F}) & \leftarrow & \text{Hom}_{\mathcal{O}_X}((j_u)_* \mathcal{O}_X, \mathcal{F}) \\
 0 & \leftarrow & \mathcal{F}(v) & \leftarrow & \mathcal{F}(u)
 \end{array}$$

Prop: (X, \mathcal{O}_X) ringed space. For an exact seq $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ in Mod \mathcal{O}_X if \mathcal{F}' is flasque.

$$\dots \rightarrow \mathcal{R}^i \Gamma(\mathcal{F}') \rightarrow \mathcal{R}^i \Gamma(\mathcal{F}) \rightarrow \mathcal{R}^i \Gamma(\mathcal{F}'') \rightarrow \dots$$

if \mathcal{F}' is flasque.

(i) Then $0 \rightarrow \Gamma(X, \mathcal{F}') \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}'') \rightarrow 0$
is exact

(ii) For a flasque sheaf of abelian groups \mathcal{G} , $H^i(X, \mathcal{G}) = 0 \quad \forall i > 0$

Pf. Given $s \in \Gamma(X, \mathcal{F}'')$, by Zorn's lemma, choose a maximal set in the non-empty set

$$\{ (U, t) \mid U \subseteq X, t \in \mathcal{F}(U) \text{ s.t. } t \text{ maps to } s|_U \},$$

call it (U_s, t_s) . If $U_s \neq X$, pick $x \in X \setminus U_s$

\exists a open nbhd V of x and $t_V \in \mathcal{F}(V)$ s.t.
 t_V maps to $s|_V$.

$$t_V|_{U_s \cap V} - t_s|_{U_s \cap V} \in \mathcal{F}'(U_s \cap V).$$

Since \mathcal{F}' is flasque, $\exists \tilde{t} \in \mathcal{F}'(X)$ mapping to the difference above. So $t_V - \tilde{t}|_V$ and t_s agree on $U_s \cap V$. Then $\exists!$ $t' \in \mathcal{F}(V \cup U_s)$ s.t.
 $t'|_{U_s} = t_s$. So $(V \cup U_s, t')$ is bigger than (U_s, t_s)
contradicting the maximality.

Then $U_s = X$.

(iii) Choose an injection $\mathcal{G} \rightarrow I$ where I is an injective \mathcal{O}_X -mod.

Show that I/\mathcal{G} is flasque by considering the diag and that $I, \mathcal{G}|_U, I|_U$ are flasque.

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{G}(X) & \rightarrow & I(X) & \rightarrow & I/\mathcal{G}(X) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{G}(U) & \rightarrow & I(U) & \rightarrow & I/\mathcal{G}(U) \rightarrow 0 \end{array}$$

Long exact sequence of sheaf cohomology implies

$$0 \rightarrow H^0(X, \mathcal{G}) \rightarrow H^0(X, I) \rightarrow H^0(X, I/\mathcal{G}) \rightarrow H^1(X, \mathcal{G}) \rightarrow H^1(X, I) \rightarrow \dots$$

$$0 \rightarrow H^0(X, \mathcal{G}) \rightarrow H^0(X, \mathcal{I}) \rightarrow H^0(X, \mathcal{I}/\mathcal{G}) \rightarrow H^1(X, \mathcal{G}) \rightarrow H^1(X, \mathcal{I}) \rightarrow H^1(X, \mathcal{I}/\mathcal{G}) \rightarrow H^2(X, \mathcal{G}) \rightarrow \dots$$

Since $H^1(X, \mathcal{I}) = 0$, (i) $\Rightarrow H^1(X, \mathcal{G}) = 0$

Moreover we have $H^{i+1}(X, \mathcal{G}) \cong H^i(X, \mathcal{I}/\mathcal{G}) \quad \forall i > 0$

By induction show that $H^i(X, \mathcal{G}) = 0 \quad \forall i > 0$.

Proof: (X, \mathcal{O}_X) ringed space, $\mathcal{F}_i \in \text{Mod } \mathcal{O}_X$.

(i) Given a flasque resolution $\mathcal{F}_i \rightarrow \mathcal{J}^i$ i.e. each \mathcal{J}^i is flasque, $\mathcal{F}_i \cong H^0(\mathcal{J}^i)$, $H^i(\mathcal{J}^i) = 0 \quad \forall i > 0$

The i -th cohomology of $\Gamma(X, \mathcal{J}^i)$ computes $H^i(X, \mathcal{F}_i)$

(ii) T.F.D.C. $\forall i \geq 0$
 sheaves of abelian groups $= \text{Mod } \mathbb{Z} \xrightarrow{H^i(X, -)}$
 \uparrow
 $\text{Mod } \mathcal{O}_X \xrightarrow{H^i(X, -)}$ abelian groups

i.e. for an \mathcal{O}_X -mod \mathcal{F}_i , the sheaf cohomology of \mathcal{F}_i obtained by resolving by injective sheaves of abelian groups and injective sheaves of \mathcal{O}_X -mods are the same.

Thm: ① A noeth ring, I be an injective mod. Then \tilde{I} is a flasque sheaf.

② On a noeth scheme any q -coh \mathcal{O}_X -mod can be resolved by a complex of q -coh flasque \mathcal{O}_X -mods.

Pf. Step 1: For an ideal $J \subseteq A$. Set
 $\Gamma_J(I) = \{x \in I \mid J^n \cdot x = 0 \text{ for some } n \in \mathbb{N}\}$
 $= \{x \in I \mid \frac{x}{1} \in I_p \text{ is zero if } p \notin V(J)\}$.

So $\Gamma_J(I)$ only depends on $V(J)$ and I .

So $\Gamma_J(I) = \Gamma_{V(J)}(I)$

Claim: $\Gamma_J(I)$ is an injective A -mod. (Lemma 3.2, Hart)

Step 2: For any $f \in A$, the map $I \rightarrow I_f$ is surjective. (Lemma 3.3, Hart)

Step 2: For any $f \in A$, the map $I \rightarrow I_f$ is surjective.
(Lemma 3.3, Hart)

Noeth induction: T noeth top space (i.e. every decreasing chain of closed subs. stabilize). Let \mathcal{P} be a property of closed subs. such that for every closed $Z \subseteq T$ if \mathcal{P} holds for every proper closed subs. of Z , \mathcal{P} holds for T . Then \mathcal{P} holds for X . [Note by our assumption \mathcal{P} holds for \emptyset]
Pf of noeth induction:

Consider the collection of closed subs. on which \mathcal{P} fails. If this set is non-empty there must be a smallest elt. But our assumption contradicts that. \square

X scheme, \mathcal{F}_X Mod \mathcal{O}_X .

Define $\text{supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$

For $s \in \mathcal{F}(U)$, $\text{supp}(s) = \{x \in U \mid sx \neq 0\}$

$X = \text{Spec}(A)$. We say that a closed subs. Z has property \mathcal{P} if for every injective A -module M with $\text{supp}(M) \subseteq Z$, \tilde{M} is flasque.

Let Z be a closed subs. s.t. every proper closed subs. has property \mathcal{P} . Take an injective mod M s.t. $\text{supp } \tilde{M} \subseteq Z$.

Given $U \subseteq_{\text{open}} X$, if $U \cap Z = \emptyset$,

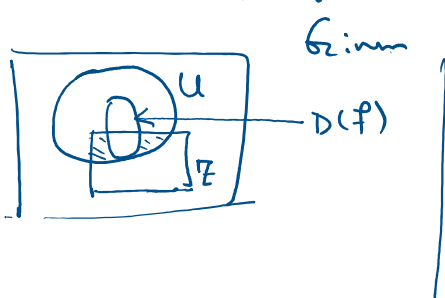
$\tilde{M}(U) \rightarrow \tilde{M}(U) = 0$ is surjective.

Assume $U \cap Z \neq \emptyset$ choose $f \in A$ s.t. $D(f) \subseteq U$ and $D(f) \cap Z \neq \emptyset$. $Z' = X - D(f) = V(f)$

Define $M' = \{x \in M \mid f^n \cdot x = 0 \text{ for some } n \text{ depending on } x\}$

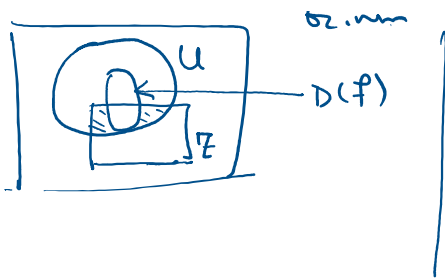
Note $\tilde{M}'(U) = \{s \in \tilde{M}(U) \mid \text{supp}(s) \subseteq U \cap Z'\}$

(*) Since $\text{supp}(M') \subseteq V(f) \cap \text{supp}(M) \subseteq V(f) \cap Z \subseteq Z$, \tilde{M}' is flasque by our induction hypothesis



$s \in \Gamma(U, \tilde{M})$, $\exists s' \in \Gamma(X, \tilde{M})$ s.t.

$$s' \mid_{D(f)} = s \mid_{D(f)}$$



$$s \in \Gamma(U, \tilde{M}), \quad \rho \in \Gamma(U, \tilde{M}')$$

$$s' \mid_{D(f)} = s \mid_{D(f)}$$

Then $s - s' \mid_U \in \Gamma(U, \tilde{M})$ has support in $V(f) \cap Z$

Then $s - s' \mid_U \in \Gamma(U, \tilde{M}') \subseteq \Gamma(U, \tilde{M})$. Since \tilde{M}' is flasque

[By our induction hypothesis, since $\overline{\text{supp}(\Gamma_Z(\tilde{M}))} \not\subseteq Z$, $\tilde{\Gamma}_Z(\tilde{M})$ is a flasque] Choose $t \in \tilde{M}'(X) = M' \subseteq M$ s.t. $t \mid_U = s - s' \mid_U$

Then $(s' + t) \mid_U = s$, $s' + t \in \Gamma(X, \tilde{M}) \cong M$.

Prop: X be a noetherian scheme. Every q -coh \mathcal{O}_X -mod \mathcal{F} admits a resolution by q -coh flasque sheaves.

Pf: Given $\mathcal{F} \in \text{Coh}(X)$, enough to produce an injection $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}$, where \mathcal{I} is a q -coh flasque \mathcal{O}_X -mod.

Choose an affine ^{open} covering $X = \bigcup_{i=1}^n U_i$.

For each i , choose an injection $0 \rightarrow \mathcal{F} \mid_{U_i} \rightarrow \mathcal{I}_i$ where \mathcal{I}_i is an injective $\mathcal{O}_X(U_i)$ -mod. Denote the immersion $U_i \rightarrow X$ by i_i

Then get an injection

$$0 \rightarrow \mathcal{F} \rightarrow \bigoplus_{j=1}^n (i_j)_* (\mathcal{F} \mid_{U_j}) \rightarrow \bigoplus_{j=1}^n (i_j)_* (\mathcal{I}_j)$$

$$\downarrow \mathcal{F}(U) \quad \downarrow (\mathcal{F} \mid_{U \cap U_j})$$

Claim: $\bigoplus_{j=1}^n (i_j)_* (\mathcal{I}_j)$ is flasque.

Pf: \mathcal{I}_j is flasque on U_j , so $(i_j)_* (\mathcal{I}_j)$ is flasque, so is the direct sum.

Thm. Let X be a noeth affine scheme. $\mathcal{F} \in \text{q.coh}(X)$.

$$H^i(X, \mathcal{F}) = 0 \quad \forall \quad i > 0.$$

Rmk. THE ABOVE THM IS TRUE WITHOUT ANY NOETH HYPOTHESIS.

Pf. Let $X = \text{Spec}(A)$, $\mathcal{F} \cong \tilde{M}$ for some $M \in \text{Mod } A$.

Choose an injection resolution $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$
 in $\text{Mod } A$.

Then have a flasque resolution in $\text{Mod } A$,

$$0 \rightarrow \tilde{M} \rightarrow \tilde{I}^0 \rightarrow \tilde{I}^1 \rightarrow \dots$$

Since flasque sheaves are $\Gamma(X, -)$ acyclic

$$\begin{aligned} H^i(X, \tilde{M}) &= H^i(\Gamma(\tilde{I}^0)) \\ &= H^i(I^0 \rightarrow I^1 \rightarrow \dots \rightarrow I^i \rightarrow I^{i+1} \rightarrow \dots) \\ &= 0 \quad \text{for } i > 0. \end{aligned}$$

[$\because I^j$ q.coh
 $\tilde{I}^j(X) \cong I^j$]

Thm. Let X be a noeth scheme. T.F.A.E

① X is affine

② \forall q.coh \mathcal{O}_X -mod \mathcal{F} , $H^i(X, \mathcal{F}) = 0 \quad \forall i > 0$.

③ For all q.coh ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, $H^i(X, \mathcal{I}) = 0 \quad \forall i > 0$.

Pf. Hart Thm 3.7.

Start of 13.12.24 Lecture

End of 11.12.24 lecture

Thm: X be a noetherian topological space.

Define $\dim(X) = \sup \{n \mid \exists \text{ a } \overset{\text{strict}}{\text{chain of } n \text{ non closed sets}} \\ X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_n\}$

For any sheaf of abelian groups \mathcal{F} on X , $H^i(X, \mathcal{F}) = 0$
 $\forall i > \dim(X)$.

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 $\forall i > \dim(X)$.

We do not prove it in class but prove a weaker version.
For a proof of the above, see Thm 2.7, Ch III Hart.