Morphism to projective space

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26.11.25 Lecture continued.

We are interested (a general version) of the following question.

Question. Let k be a field. X be a finite type scheme $/k$. When is X quasi-projective, i.e. when is X isom to an open subscheme of a closed subscheme of $\mathbb{P}^n_k$.

Notation and Terminology:

$\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[x_0, \dots, x_n])$, For any scheme Y

$\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} Y$, $\mathcal{O}_{\mathbb{P}^n_Y}(1) = j^* \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$.

$$\begin{tikzcd} \mathbb{P}^n_Y \arrow[r, "j"] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ Y \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

$\mathbb{P}^n_{\operatorname{Spec} A} =: \mathbb{P}^n_A$,

$\mathcal{L}$ invertible sheaf on X, $s \in \Gamma(X, \mathcal{L})$.

$D_s = \{ x \in X \mid s_x \notin m_x \mathcal{L}_x \}$. $D_s \subseteq X$ open.

Note. The map $\mathcal{O}_X \longrightarrow \mathcal{L}$, $1 \longmapsto s$

is an isom on D_s.

Def. For any $t \in \Gamma(V, \mathcal{L})$, $V \supseteq D_s$, $t/s \in \mathcal{O}_X(U)$ is the unique elt s.t $t_{|U} = t/s \cdot s_{|U}$

§ Morphism to $\mathbb{P}^n$

Let A be a ring.

Thm. Let X be an A-scheme (i.e fix a morphism $X \to \operatorname{Spec} A$) $\mathcal{L}$ be an invertible $\mathcal{O}_X$-mod. $s_0, \dots s_n \in \Gamma(X, \mathcal{L})$ s.t $X = \bigcup_{i=0}^{n} D_{s_i}$.

- There is a A-scheme morphism
$$\varphi : X \longrightarrow \mathbb{P}^n_A$$
and an isom $\theta : \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \xrightarrow{\sim} \mathcal{L}$ which sends $\varphi^*(x_i)$ to s_i

- Any A-scheme morphism
$$\psi : X \longrightarrow \mathbb{P}^n_A$$
such that there is an isom $\psi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \longrightarrow \mathcal{L}$ sending $\psi^*(x_i)$ to s_i must coincide with φ.

Rmk. Given $f : X_1 \longrightarrow X_2$ scheme map, $\mathcal{F}_2 \in \operatorname{Mod}_{\mathcal{O}_{X_2}}$
$t \in \Gamma(X_2, \mathcal{F}_2)$. Recall $f^* \mathcal{F}_2 := f^{-1}\mathcal{F}_2 \otimes \mathcal{O}_{X_1}$
t gives a section $f^{-1}(t) \in \Gamma(X_1, f^{-1}(\mathcal{F}_2))$
$f^*(t)$ denotes the section $f^{-1}(t) \otimes 1$.

More canonically, the adjunction between f^*, f_* gives a map (image of $\mathrm{id} \in \operatorname{Hom}(f^*\mathcal{F}_2, f^*\mathcal{F}_2)$), $\mathcal{F}_2 \longrightarrow f_*(f^*\mathcal{F}_2)$. f^*t is the image of t under this map.

Pf of Thm.

$$\mathbb{P}^n_A = \bigcup_{i=0}^{n} D_{x_i}, \quad D_{x_i} = D_+(x_i) = \operatorname{Spec}\Big(A\Big[\tfrac{x_0}{x_i}, \dots \underbrace{\tfrac{x_i}{x_i}}_{\text{omitted}}, \dots \tfrac{x_n}{x_i}\Big]\Big)$$

There is a unique A-scheme map φ_i for each i

$$X \supseteq D_{s_i} \longrightarrow D_{x_i}$$

induced by the A-alg map

$$A\Big[\tfrac{x_0}{x_i} \dots \widehat{x_i/x_i}, \quad x_n/x_i\Big] \longrightarrow \Gamma(D_{s_i}, \mathcal{O}_X)$$
$$x_j/x_i \longmapsto s_j/s_i$$

$\varphi_i|_{D_{s_i} \cap D_{s_j}}, \varphi_j|_{D_{s_i} \cap D_{s_j}} : D_{s_i} \cap D_{s_j} \longrightarrow D_{x_i} \cap D_{x_j}$

both correspond to the same A-alg map

$$A\Big[\tfrac{x_0}{x_i}, x_1/x_i, \dots, x_i/x_i \cdot x_n/x_i\Big]\Big[(x_j/x_i)^{-1}\Big] \longrightarrow \Gamma(D_{s_i} \cap D_{s_j}, \mathcal{O}_X)$$
$$\|$$
$$A\Big[\tfrac{x_0}{\ } \quad , x_n/\ \Big]\Big[\big(\tfrac{x_i}{\ }\big)^{-1}\Big]$$

$$A\left[\frac{x_0}{x_i}, x_1/x_i, \dots, x_j/x_i, \dots x_n/x_i\right]\left[(x_j/x_i)^{-1}\right] \longrightarrow \Gamma(D_{s_i} \cap D_{s_j}, \mathcal{O}_X)$$
$$\parallel$$
$$A\left[\frac{x_0}{x_j}, \dots, x_n/x_j\right]\left[\left(\frac{x_i}{x_j}\right)^{-1}\right]$$

- Thus $\varphi_i|_{D_{s_i}\cap D_{s_j}} = \varphi_j|_{D_{s_i}\cap D_{s_j}}$
- Thus $\exists!\ \varphi : X \to \mathbb{P}^n_A$ such that $\varphi|_{D_{s_i}} = \varphi_i$.

The desired isom $\varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \to \mathcal{L}$ is constructed as follows. Note $\varphi^{-1}(D_{x_i}) = D_{s_i}$ by construction.

$$\theta_i : \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1))|_{D_{s_i}} \xrightarrow{\sim} \mathcal{O}_{D_{s_i}}\cdot\varphi^*(x_i) \longrightarrow \mathcal{O}_{D_{s_i}}\cdot s_i \xrightarrow{\sim} \mathcal{L}|_{D_{s_i}}$$
$$\varphi^*(x_i)|_{D_{s_i}} \longmapsto s_i|_{D_{s_i}}$$
$$\downarrow\wr \qquad\qquad \uparrow\wr$$
$$\varphi_i^*(\mathcal{O}_{\mathbb{P}^n_A}(1)|_{D_{x_i}}) \xrightarrow{\sim} \varphi_i^*(\mathcal{O}_{\mathbb{P}^n_A|_{D_{x_i}}}\cdot x_i)$$

Since $\theta_i|_{D_{s_i}\cap D_{s_j}} = \theta_j|_{D_{s_i}\cap D_{s_j}}$

$\exists!\ \theta : \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \longrightarrow \mathcal{L}$ s.t. $\theta|_{D_{s_i}} = \theta_i$

Clearly $\theta(\varphi^*(x_i)) = s_i\ \forall i$. $\left(\theta_j(\varphi^*(x_i)) = \theta_j\left(\frac{s_i}{s_j}\varphi^*(x_j)\right)\right.$
$= s_i/s_j \cdot s_j = s_i$

For the uniqueness, given such a ψ,
The existence of the isom
$\psi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \xrightarrow{\sim} \mathcal{L}$ sending $\psi^*(x_i)$ to s_i
implies $\psi^{-1}(D_{x_i}) = D_{s_i}$. Indeed for $x \in X$, we have an isom

$$k(x) \otimes_{k(\psi(x))} \frac{\mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)}}{m_{\psi(x)}\mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)}} \xrightarrow{\sim} \frac{\mathcal{L}_x}{m_x\mathcal{L}_x}$$
$$1 \otimes x_i \,(\mathrm{mod}\ m_{\psi(x)}\mathcal{O}_{\mathbb{P}^n}(1)_{\psi(x)}) \longmapsto s_i\ (\mathrm{mod}\ m_x\mathcal{L}_x)$$

$x \in X - D_{s_i} \iff s_i\ (\mathrm{mod}\ m_x\mathcal{L}_x) = 0 \iff 1 \otimes x_i\ (\mathrm{mod}\ m_{\psi(x)}\mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)}) = 0$
since $k(x) \hookleftarrow k(\psi(x))$ is a field extension the last quantity is zero
iff $x_i\ (\mathrm{mod}\ m_{\psi(x)}\mathcal{O}_{\mathbb{P}^n_A}(1)) = 0 \iff \psi(x) \in \mathbb{P}^n_A - D_{x_i}$

That map $A[x_0/x_i, \dots, x_n/x_i] \to \Gamma(D_{x_i}, \mathcal{O}_X)$
that induces $\psi|_{D_{x_i}}$ sends x_j/x_i to s_j/s_i:

On D_{x_i} $\psi^*(x_j) = \psi^\#(x_j/x_i)\cdot\psi^*(x_i)$

So $\psi^\#(x_j/x_i) = \psi^*(x_j)/\psi^*(x_i)$
$= s_j/s_i$

Thus $\psi|_{D_{s_i}} = \varphi|_{D_{s_i}}\ \forall i$. ∎ End of 22.11.24 lecture

Rmk When X is f.t over an algebraically closed field.
The map induced by $s_0 \dots s_n$ sends
a closed pt $x \in X$ to $[s_0(x) : s_1(x) : \quad : s_n(x)] \in \mathbb{P}^n_k$.

Thm Given an A-scheme X and $n \in \mathbb{N}$. There is a one to one correspondence

$$\frac{\left\{\mathcal{L},\ \text{ordered tuple } (s_0, \dots, s_n) \;\middle|\; \mathcal{L} \text{ invertible},\ s_i \in \Gamma(X, \mathcal{L}) \text{ s.t } \bigcup_{i=0}^n D_{s_i} = X\right\}}{\sim} \longrightarrow \left\{\begin{array}{c}A\text{-scheme map} \\ X \to \mathbb{P}^n_A\end{array}\right\}$$

where $(\mathcal{L}_1, (s_0 \dots s_n)) \sim (\mathcal{L}_2, (t_0 \dots t_n))$
iff $\exists$ an isom of $\mathcal{O}_X$-mods
$\theta : \mathcal{L}_1 \to \mathcal{L}_2$ sending s_i to $t_i\ \forall i$.

Pf. Given $\varphi : X \to \mathbb{P}^n_A$ of A-schemes take
$\mathcal{L}_\varphi = \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1))$ and $s_i = \varphi^*(x_i)$.

The morphism induced by $(\mathcal{L}_\varphi, (s_0 \dots, s_n))$ is indeed φ. The one-one correspondence is H.W.

Eg: $k = \mathbb{C}, \overline{\mathbb{Q}}, \dots$ $X = \mathbb{P}^1_k$, $\mathcal{L} = \mathcal{O}(2)$
$\Gamma(\mathbb{P}^1, \mathcal{O}(2)) = k x^2 \oplus kxy \oplus ky^2$.

get $\mathbb{P}^1_k \longrightarrow \mathbb{P}^2_k$
$[x_0 : x_1] \longrightarrow [x_0^2 : x_0 x_1 : x_1^2]$
The image is $V(Y^2 - XZ)$

End of 26.11.25 lecture

§ Criterion for having an embedding

Recall. A locally closed subspace of a topological space is an intersection of a closed and open subset with the induced topology.

Recall: A locally closed subspace of a topological space is an intersection of a closed and open subset with the induced topology.

Def. • A morphism of schemes $\varphi: X \to Y$ is called a locally closed immersion iff (1) φ induces a homeo from X to a locally closed subset of Y and (2) $\mathcal{O}_{Y,y} \to (\varphi_* \mathcal{O}_X)_y$ is surjective at all points of $y \in \varphi(X)$ — Locally closed immersion

• $\varphi: X \to Y$ is called a closed immersion if $\varphi(X) \subseteq Y$ is closed.

• $\varphi: X \to Y$ is called an open immersion if $\varphi(X) \subseteq Y$ is open and $\forall\ y \in \varphi(X)$, $\mathcal{O}_{Y,y} \to (\varphi_* \mathcal{O}_X)_y$ is an isom.

Prop. Every locally closed immersion $\varphi: X \to Y$ can be factored as
1) $X \xrightarrow{i} U \xrightarrow{j} Y$ where j is an open immersion and i is a closed immersion.

Pf. Write $\varphi(X) = Z \cap U$ where $Z \underset{\text{closed}}{\subseteq} Y$, $U \underset{\text{open}}{\subseteq} Y$.
Then φ factors as $X \xrightarrow{i} U \xrightarrow{j} Y$ as a scheme map, where U has the induced open scheme structure and j is the open immersion.
We claim that i is a closed embedding. Indeed i is a homeo onto the image and $i(X) = Z \cap U$ is closed in U.
Note that for a point in U which is not in $i(X)$, the stalk $(i_* \mathcal{O}_X)_y = \varphi_* (\mathcal{O}_X)_y = 0$. Indeed one can choose an open nbhd V of y such that $V \cap i(X) = \phi$. Then $i^{-1}(V) = \phi$. So $(i_* \mathcal{O}_X)_y = 0$.
We claim that the induced map
$$\mathcal{O}_U \to i_* \mathcal{O}_X \text{ is surjective.}$$
If $y \in i(X)$, $\mathcal{O}_{U,y} \to (i_* \mathcal{O}_X)_y$ is the same as the induced map $\mathcal{O}_{X,y} \to (\varphi_* \mathcal{O}_X)_y$ which is surjective by our assumption.
If $y \notin i(X)$, $\mathcal{O}_{U,y} \to (i_* \mathcal{O}_X)_y$ is surjective because the target is zero.

Rmk. In general it is not true that any locally closed immersion $\varphi: X \to Y$ can be factored as $X \xrightarrow{j} Z \xrightarrow{i} Y$ where j is an open immersion and i is a closed immersion.
However, we have the following.

Prop. Let $\varphi: X \to Y$ be a locally closed immersion such that $\varphi_* \mathcal{O}_X$ is quasi-coherent. Then $\varphi: X \to Y$ can be factored as
$$X \xrightarrow{j} Z \xrightarrow{i} Y$$
where j is an open immersion and i is closed.

Pf. Let $\mathcal{I} = \ker(\mathcal{O}_Y \to \varphi_* \mathcal{O}_X)$. For $y \in \varphi(X)$, since $\mathcal{O}_{Y,y} \to (\varphi_* \mathcal{O}_X)_y$ is surj, $\mathcal{I}_y \subseteq m_y$.
Since $\varphi_* \mathcal{O}_X$ is q.coh, $\mathcal{I}$ is q.coh.
So $V(\mathcal{I}) = \{ y \in Y \mid \mathcal{I}_y \subseteq m_y \}$ is closed in Y
$V(\mathcal{I}) \supseteq \varphi(X)$. Equip $V(\mathcal{I})$ with the scheme structure given by $\mathcal{O}_Y/\mathcal{I}$, call the scheme Z.
Note $\varphi(X)$ is open in $V(\mathcal{I})$. Since $\mathcal{I} \cdot \varphi_* \mathcal{O}_X = 0$, the map $\varphi: X \to Y$ factors through the closed immersion $\varphi: X \xrightarrow{j} Z \xrightarrow{i} Y$, i is the closed immersion.
We claim j is an open immersion. Indeed what remains to check is that $\forall\ y \in \varphi(X) \subseteq Z$, the induced map $\mathcal{O}_{Z,y} \to (j_* \mathcal{O}_X)_y$ is an isom. But $\mathcal{O}_{Z,y} = \dfrac{\mathcal{O}_{Y,y}}{\mathcal{I}_y}$ is isom to $(j_* \mathcal{O}_X)_y$ for $y \in \varphi(X)$ via the map induced by φ and hence by j. □

Def. Given a scheme Y. Sch_Y is the category
• whose objects are schemes X together with a morphism $X \to Y$
• Given $X_1 \to Y$, $X_2 \to Y$ in Sch_Y a Y-scheme morphism is a scheme map $\varphi: X_1 \to X_2$ s.t
$$\begin{tikzcd} X_1 \arrow[rr, "\varphi"] \arrow[dr] & & X_2 \arrow[dl] \\ & Y & \end{tikzcd}$$
commutes.

Def. • Let X be a Y scheme. An invertible $\mathcal{O}_X$ mod $\mathcal{L}$ is called Y-very ample if

$\searrow_Y \swarrow$

<u>Def.</u> • Let X be a Y scheme. An invertible $\mathcal{O}_X$ mod $\mathcal{L}$ is called <u>Y-very ample</u> if $X \to Y$ can be factored as

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbb{P}^n_Y \\ & \searrow \; \swarrow & \\ & Y & \end{array} \qquad *$$

where φ is locally closed and $\mathcal{L} \xrightarrow{\sim} \varphi^*(\mathcal{O}_{\mathbb{P}^n_Y}(1))$.

• Whenever the structure map $X \to Y$ has a factorization as in $*$, we say <u>X is quasi projective over Y</u>.

• Whenever the structure map $X \to Y$ can be factored as in $*$ with φ being a closed embedding, the morphism $X \to Y$ is called projective.

<u>Prop.</u> A ring. Let X/A be proper. Any locally closed immersion $\varphi : X \to \mathbb{P}^n_A$ is a closed immersion. Thus X is projective $/A$ $\Leftrightarrow$ X is quasi-projective $/A$.

<u>Pf.</u> $\mathbb{P}^n_A$ is separated, X/A proper. So $\varphi(X)$ is closed in $\mathbb{P}^n_A$. ∎

<u>Thm.</u> Let X be a finite type scheme $/A$ = noetherian; $\mathcal{L}$ be an invertible sheaf on X.
$\mathcal{L}$ is ample $\Leftrightarrow$ $\mathcal{L}^n$ is very ample for some n.

<u>Pf.</u> In our set up a very ample invertible sheaf $\mathcal{L}'$ is ample.
Indeed, choose a locally closed embedding $X \xrightarrow[\varphi]{} \mathbb{P}^n_A$ s.t $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1) \xrightarrow{\sim} \mathcal{L}'$.
Factor φ as $X \xrightarrow[i]{} U \xrightarrow[j]{} \mathbb{P}^n_A$, where i is a closed immersion and j is an open immersion. Let $\mathcal{F} \in \mathrm{Coh}(X)$. Since i is a closed immersion $i_*\mathcal{F} \in \mathrm{Coh}(U)$ and $i^*(i_*\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$. Recall $\exists\ \mathcal{G} \in \mathrm{Coh}(\mathbb{P}^n_A)$ s.t $j^*\mathcal{G} \xrightarrow{\sim} i_*\mathcal{F}$.
Recall that $\mathcal{O}_{\mathbb{P}^n_A}(1)$ is ample. So $\exists\ n_0$ s.t $\forall n \geq n_0$ $\mathcal{G} \otimes \mathcal{O}_{\mathbb{P}^n_A}(1)^n$ is glbl gen
So $\varphi^*(\mathcal{G} \otimes \mathcal{O}_{\mathbb{P}^n_A}(1)^{\otimes n}) \xrightarrow{\sim} \varphi^*(\mathcal{G}) \otimes \mathcal{L}^n \xrightarrow{\sim} \mathcal{F} \otimes \mathcal{L}^n$ is glbl gen $\forall n \geq n_0$.

$\mathcal{L}^m$ very ample $\Rightarrow$ $\mathcal{L}^m$ ample $\Rightarrow$ $\mathcal{L}$ ample.

Now assume $\mathcal{L}$ is ample.
<u>Step 1:</u> For each $x \in X$, there exists $s_x \in \Gamma(X, \mathcal{L}^{n_x})$ such that D_{s_x} is an affine open nbhd of x.

<u>Pf. of Step 1:</u> Fix $x \in X$ and an affine open nbhd U_x of x s.t $\mathcal{L}|_{U_x} \cong \mathcal{O}_{U_x}$. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf s.t $V(\mathcal{I}) = X - U_x$. Choose n_x s.t $\mathcal{I} \otimes \mathcal{L}^n$ is globally generated for $n \geq n_x$. So there exists $s_x \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{n_x}) \subseteq \Gamma(X, \mathcal{L}^{n_x})$
$s_x \notin m_x(\mathcal{I} \otimes \mathcal{L}^{n_x}) = m_x \mathcal{L}^{n_x}$ [$\because$ $\mathcal{I}_x = \mathcal{O}_{X,x}$]
Note $\forall\ y \in X - U_x$, $(s_x)_y \in \mathcal{I}_y \otimes \mathcal{L}^{n_x}_y \subseteq m_y \mathcal{L}^{n_x}_y$

So $D_{s_x} \subseteq U_x$. Choose an isom $\mathcal{L}|_{U_x} \xrightarrow{\sim} \mathcal{O}_{U_x}$. Assume s_x goes to f_x
$D_{s_x} = D_{f_x}$ in U_x. So D_{s_x} is affine. ∎

Since X is quasi-compact, there is a finite covering
$$X = \bigcup_{i=1}^{r} D_{s_{x_i}}$$
Replace $\mathcal{L}$ by $\mathcal{L}^{n_{x_1} n_{x_2} \cdots n_{x_r}}$ and s_{x_i} by $s_{x_i}^{n_{x_1} \cdots n_{x_r}/n_{x_i}}$

Thus $\exists\ s_{x_1} \cdots s_{x_r} \in \Gamma(X, \mathcal{L})$ such that
$$X = \bigcup_{i=1}^{r} D_{s_{x_i}} \quad \text{and each } D_{s_{x_i}} \text{ is affine}$$
For each $1 \leq i \leq r$, make a choice
$$\Gamma(D_{s_{x_i}}, X) = A[y_{ij} \mid 1 \leq j \leq n_i]$$
$\exists\ l \in \mathbb{N}$ s.t $y_{ij} \otimes s_{x_i}^l$ extends to a global section t_{ij} of $\Gamma(X, \mathcal{L}^l)$, $\forall\ 1 \leq i \leq r$, $1 \leq j \leq n_i$

Consider the morphism φ to some $\mathbb{P}^N_A$ given by $\mathcal{L}, \{t_{ij}, s^{\ell}_{x_i}\}_{1 \le i \le r, 1 \le j \le n_i}$. $N = n_1 + n_2 + \cdots + n_r + r - 1$

$\mathbb{P}^N = \mathrm{Proj}\,(A[x_{ij}, y_i \mid 1 \le i \le r, 1 \le j \le n_i])$

Claim: φ is a locally closed immersion

Pf. Since $X = \bigcup_{i=1}^{r} D_{s^{\ell}_{x_i}}$, φ factors through $\bigcup_{i=1}^{r} D_{y_i} = V$

We show $\varphi: X \longrightarrow V \subseteq \mathbb{P}^N$ is a closed immersion.

For that, enough to show

$D_{s_{x_i}} = \varphi^{-1}(D_{y_i}) \longrightarrow D_{y_i}$ is a closed immersion $\forall i$

$\Longleftrightarrow$ $\Gamma(D_{y_i}, \mathbb{P}^N) \longrightarrow \Gamma(D_{s_{x_i}}, X)$ is surjective $\forall i$

$$A[y_{ij} \mid 1 \le j \le r]$$

$\frac{t_{ij}}{s^{\ell}_{x_i}} \longmapsto y_{ij}$ [$\because$ t_{ij} extends $y_{ij} \otimes s^{\ell}_{x_i}$]. $\square$

Prop. (i) A noetherian ring. $X \in \mathrm{Sch}_A$ is quasi-projective iff X has an ample invertible sheaf.
(ii) If X is proper$/A$. Then X is projective $\Longleftrightarrow$ X has an ample invertible sheaf.

§ Examples of projective morphisms. Blow-ups.

- X noetherian scheme, $\mathcal{S} = \bigoplus_{n \in \mathbb{N}} \mathcal{S}_n$ be an $\mathbb{N}$-graded quasi-coherent sheaf of $\mathcal{O}_X$-algebras such that
  - Each $\mathcal{S}_n$ is a q.coh $\mathcal{O}_X$-submod of $\mathcal{S}$
  - The $\mathcal{O}_X$-alg structure on $\mathcal{S}$, comes from a ring map $\mathcal{O}_X \to \mathcal{S}_0$
- for each affine $\mathrm{Spec}\,(A) \subseteq X$
  $\mathcal{S}(\mathrm{Spec}\, A) = (S_0 \oplus S_1 \oplus S_2 \oplus \cdots)^{\sim}$
  where $\oplus S_i$ is standard graded.

Prop. • There exists a scheme denoted $\underline{\mathrm{Proj}}_X(\mathcal{S})$ with a map $\pi: \underline{\mathrm{Proj}}_X(\mathcal{S}) \to X$ such that for each affine open $\mathrm{Spec}(A) \subseteq X$, $\pi^{-1}(U) \cong \mathrm{Proj}\,(S_0 \oplus S_1 \oplus \cdots \oplus)$ where $\mathcal{S}(\mathrm{Spec}\, A) = (\oplus S_i)^{\sim}$ (see page 166 Hart).

- There exists an invertible sheaf denoted $\mathcal{O}(1)$ on $\underline{\mathrm{Proj}}_X(\mathcal{S})$ s.t $\mathcal{O}(1)|_{\mathrm{Spec}(A)} \cong \tilde{S}(1)$. where $S = \oplus S_j$.

- Given an invertible sheaf $\mathcal{L}$ and $\mathcal{S}$ as above define $\mathcal{S}' = \oplus \mathcal{S}_n \otimes \mathcal{L}^n$
  Then there is a natural isom
  $\varphi: \underline{\mathrm{Proj}}\,(\mathcal{S}') \longrightarrow \underline{\mathrm{Proj}}\,(\mathcal{S})$ such that $\mathcal{O}(1) \cong \varphi^* \mathcal{O}(1) \otimes \pi^*(\mathcal{L})$ where
  $\pi: \underline{\mathrm{Proj}}\,(\mathcal{S}) \to X$ is the natural projection map.

eg. $\mathcal{E} = \bigoplus^{r+1} \mathcal{O}_X$ $\quad r \in \mathbb{N}$
$\mathcal{S} = \mathrm{Sym}^{\bullet}(\mathcal{E}) = \mathcal{O}_X \oplus \mathrm{Sym}^1 \mathcal{O}_X^{\oplus r+1} \oplus \mathrm{Sym}^2 \cdots$
On $\mathrm{Spec}\, A$. $\mathcal{S}|_{\mathrm{Spec}\, A} \cong A[T_0, \ldots, T_r]$
"So" $\underline{\mathrm{Proj}}\,(\mathcal{S}) = X \times_A \mathbb{P}^n_A$.

Prop. If X has an ample invertible sheaf $\mathcal{L}$, and $\mathcal{S}$ is Correct. Then $\pi: \underline{\mathrm{Proj}}\,(\mathcal{S}) \to X$ is projective.

Pf. Choose $n \in \mathbb{N}$ s.t $\mathcal{S}_1 \otimes \mathcal{L}^n$ is glob gen.
take $\mathcal{L}' = \mathcal{L}^n$.
Choose a surjection
$$\bigoplus^{r+1} \mathcal{O}_x \longrightarrow \mathcal{S}_1 \otimes \mathcal{L}'$$
This gives an $\mathcal{O}_X$-lin surjection $\underline{\mathrm{Sym}}^{\bullet}(\mathcal{O}_x^{r+1}) \to \oplus \mathcal{S}_n \otimes \mathcal{L}'^n = \mathcal{S}'$

and thus a closed embedding of sch$_X$.

$$\begin{tikzcd} \underline{\mathrm{Proj}}\,(\mathcal{S}') \arrow[r, hook] \arrow[dr] & \underline{\mathrm{Proj}}\,(\underline{\mathrm{Sym}}^{\bullet}(\mathcal{O}_x^{r+1})) = X \times \mathbb{P}^r \arrow[d] \\ & X \end{tikzcd}$$

But $\underline{\mathrm{Proj}}\,(\mathcal{S}') = \underline{\mathrm{Proj}}\,(\mathcal{S})$.

Def. Given an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. The blow up of X along $\mathcal{I}$ is the X-scheme
$\pi: \underline{\mathrm{Proj}}\,(\bigoplus_{n \in \mathbb{N}} \mathcal{I}^n) \longrightarrow X$. set $\mathrm{Bl}_{\mathcal{I}}(X) = \underline{\mathrm{Proj}}\,(\bigoplus_{n \in \mathbb{N}} \mathcal{I}^n)$
$\bigoplus_{n \in \mathbb{N}} \mathcal{I}^n = (\mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots)$

Notation: Given $f: Y_1 \to Y_2$ and an ideal sheaf

$(\mathcal{O}_X \oplus \mathcal{J}'' \oplus \mathcal{J}^2 \oplus \dots)$ $\quad - \; n \in \mathbb{N}$

<u>Notation:</u> Given $f : Y_1 \to Y_2$ and an ideal sheaf $\mathcal{J}_2 \subseteq \mathcal{O}_{Y_2}$

$$f^{-1}(\mathcal{J}_2)\mathcal{O}_{Y_1} = \mathcal{J}_2 \mathcal{O}_{Y_1}$$

is the ideal sheaf obtained as the image of the map $f^*(\mathcal{J}_2) \to \mathcal{O}_{Y_2}$

<u>Thm.</u> Let X be a noetherian scheme

(i) Given an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$, consider the blow-up map
$$\pi : Bl_{\mathcal{J}}(X) \to X$$
Then $\pi^{-1}(\mathcal{J})\,\mathcal{O}_{Bl_{\mathcal{J}}(X)} \xrightarrow{\sim} \mathcal{O}(1)$

In particular $\mathcal{J}\, Bl_{\mathcal{J}}(X)$ is invertible.

(ii) $\pi^{-1}(X \setminus V(\mathcal{J})) \to X \setminus V(\mathcal{J})$ is an isom

(iii) Let $g : Z \to X$ be a scheme map such that $\mathcal{J}\mathcal{O}_Z$ is invertible. Then f uniquely factors as

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[dr, "g"'] & Bl_{\mathcal{J}}(X) \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

End of 28.11.25 Lecture

Start of 03.12.25 Lecture

<u>Pf</u> We prove only (iii). First, we prove existence of the factorization.

W.L.O.G. X is affine. Assume $X = \operatorname{Spec} R$.
and $\mathcal{J} = \tilde{I}$, where $I = \langle f_0 \dots f_n \rangle$.

We have a surjection of graded R-algbs
$$R[x_0 \dots x_n] \to R \oplus I \oplus I^2 \oplus \dots ,$$
$$x_i \longmapsto f_i$$
where on the left $\deg x_i = 1$, $\deg R = 0$. A homogeneous polynomial $F(x_0, \dots x_n)$ is mapped to zero $\Leftrightarrow F(f_0, \dots f_n) = 0$ in R.

The above ring map gives an embedding of R-schemes
$\varphi : Bl_{\mathcal{J}}(X) \hookrightarrow \mathbb{P}^n_R$, where $\varphi^* \mathcal{O}_{\mathbb{P}^n_R}(1) \xrightarrow{\sim} \pi^{-1}\mathcal{J}\,\mathcal{O}_{Bl_{\mathcal{J}}}(X)$
isom sends $\varphi^*(x_i)$ to $\pi^{-1}(f_i) \in \pi^{-1}\mathcal{J}\, Bl_{\mathcal{J}}(X)$.

Consider $h : Z \to \mathbb{P}^n_R$ given by $g^{-1}(f_i) \in g^{-1}\mathcal{J}\mathcal{O}_Z$. h is 'defined' everywhere on Z because f_i's generate I and hence $g^{-1}(f_i)$'s generate $g^{-1}\mathcal{J}\mathcal{O}_Z$. We want to argue that h factors through $\varphi(Bl_{\mathcal{J}}(X))$. In this direction, we use the following lemma, whose proof is left as an exercise.

<u>Lemma:</u> Let A be a noeth ring, $A[x_0 \dots x_n]$ has graded str s.t $\deg(A) = 0$. $\deg(x_i) = 1$; $J \subseteq A[x_0 \dots x_n]$ be a homogeneous ideal. Let $Y = \operatorname{Proj}\left(\frac{A[x_0 \dots x_n]}{J}\right)$. An A-scheme map $\psi : S \to \mathbb{P}^n_A$ factors through $Y \Leftrightarrow$ For any homogeneous $F \in J$
$\psi^*(F(x_0 \dots x_n) \in \mathcal{O}_{\mathbb{P}^n}(\deg F)(\mathbb{P}^n_A)) = 0 \in \psi^*(\mathcal{O}_{\mathbb{P}^n}(\deg F))(S)$. ☒

Back to the proof of Thm; We need to check that for any homogeneous F s.t $F(f_0, \dots, f_n) = 0 \in R$,
$F(g^{-1}(f_0), \dots, g^{-1}(f_n)) = 0$ — which is a tautology.

Now, we prove uniqueness. Suppose, we have a commutative diag

$F(g^{-1}(f_0) \dots g^{-1}(f_n)) = 0$ — when ...

Now, we prove uniqueness. Suppose, we have a commutative diag

$$\begin{tikzcd} Z \arrow[r, "\psi"] \arrow[dr, "g"'] & Bl_{\mathcal{I}}(X) \arrow[r, hook, "\varphi"'] \arrow[d, "\pi"] & \mathbb{P}^n_R \\ & X & \end{tikzcd}$$

Recall that $\varphi^*(\mathcal{O}(1)) \xrightarrow{\sim} \pi^{-1}(\mathcal{I})\mathcal{O}_{Bl_{\mathcal{I}}(X)}$. Note that

$\psi^{-1}(\pi^{-1}\mathcal{I}\mathcal{O}_{Bl_{\mathcal{I}}(X)})\mathcal{O}_Z \xrightarrow{\sim} \psi^*(\pi^{-1}\mathcal{I}\, Bl_{\mathcal{I}}(X)) \xrightarrow{\sim} g^{-1}(\mathcal{I})\mathcal{O}_Z$.

as $\pi^{-1}\mathcal{I}\, Bl_{\mathcal{I}}(X)$ is invertible — as $g = \pi \cdot \psi$.

Now, note that $\psi: Z \to \mathbb{P}^n_R$ is determined by $\psi^*(x_0), \dots, \psi^*(x_n) \in \Gamma(Z, \psi^*(\mathcal{O}(1)))$.

But $\psi^*\mathcal{O}(1) \xrightarrow{\sim} g^{-1}(\mathcal{I})\mathcal{O}_Z$, from the argument above. The isomorphism sends $\psi^*(x_i)$ to $g^{-1}(f_i)$. This proves uniqueness of ψ.

§ Resolving indeterminacy by blowing up.

Thm. Let A be a noetherian ring, X be a noetherian A-scheme, $\mathcal{L}$ be an invertible $\mathcal{O}_X$-mod. Let $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ $(n \geq 1)$ be non-zero sections. Let $U = \bigcup_{i=0}^{n} D_{s_i}$. Then $\exists$ a coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ s.t. $V(\mathcal{I}) \subseteq X \setminus U$ and $\varphi: Bl_{\mathcal{I}}(X) \to \mathbb{P}^n_A$ such that the following diag commutes

$$\begin{tikzcd} Bl_{\mathcal{I}}(X) \supseteq \pi^{-1}(X \setminus U) \arrow[r, "\varphi"] \arrow[d, "\pi", "\wr"'] & \mathbb{P}^n_A \\ U \arrow[ur, "{[s_0 : \dots : s_n]}"'] & \end{tikzcd}$$

Pf. Note that there is a coherent ideal sheaf $\mathcal{I}$ such that the image of

$$\mathcal{O}_X^{\oplus n+1} \to \mathcal{L} \qquad \text{———} \ *$$
$$e_i \mapsto s_i$$

is $\mathcal{I}\mathcal{L}$.

Since the above map is surjective on U, $V(\mathcal{I}) \subseteq X - U$

Let $Y = Bl_{\mathcal{I}}(X) \xrightarrow{\pi} X$.

By pulling back $*$, we get a surjective map of $\mathcal{O}_Y$-mods

$$\bigoplus^{n+1} \mathcal{O}_Y \to \pi^*(\mathcal{I}\mathcal{L}) \qquad e_i \mapsto \pi^*(s_i)$$

Check that $\pi^*(\mathcal{I}\mathcal{L}) \xrightarrow{\sim} \pi^{-1}\mathcal{I}\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \pi^*\mathcal{L}$, Thus it is invertible

... The map given by $\pi^*(s_0), \dots, \pi^*(s_n)$ ...

invertible

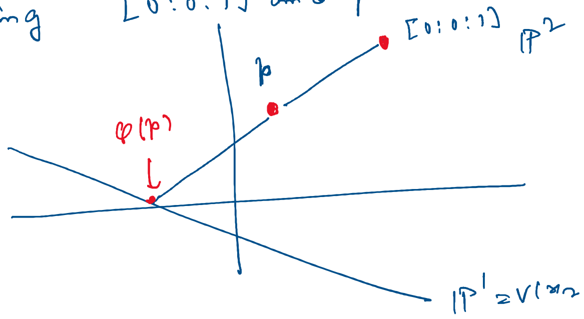
Take φ to be the map given by $\pi^*(s_0), \ldots, \pi^*(s_n) \in \Gamma(Y, \pi^*(\mathcal{L}))$

Commutativity of the diag follows easily.

<u>Eg:</u> Let k be a field. Consider the 'map' given by $x_0, x_1 \in \Gamma(\mathbb{P}^2_k, \mathcal{O}(1))$. $D_{x_0} \cup D_{x_1} = \mathbb{P}^2 - [0:0:1]$

So, we have a map $\varphi : \mathbb{P}^2 - [0:0:1] \longrightarrow \mathbb{P}^1$
$$[x_0 : x_1 : x_2] \longrightarrow [x_0 : x_1]$$

Note that $V(x_2) \cong \mathbb{P}^1_k \subseteq \mathbb{P}^2$. Check that $\varphi : \mathbb{P}^2 - [0:0:1] \to V(x_2) = \mathbb{P}^1$ follows: For a closed pt $p \in \mathbb{P}^2 - [0:0:1]$, take the line joining $[0:0:1]$ and p. This line intersects $V(x_2)$ at $\varphi(p)$



Let $\pi : Bl_{[0:0:1]} \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ be the blow up map.
$\pi^{-1}([0:0:1])$ parametrizes the tangent directions at $[0:0:1]$.
I want to explain an <u>intuitive</u> reason for why adding a point for each tangent direction is necessary to extend φ.
For each tangent direction t at $[0:0:1]$, which is a line through $[0:0:1]$ take any sequence of pts on $t - [0:0:1]$. Then $\varphi(x_n, t)$ is constant. If we change t, $\varphi(x_n, t)$ changes. So any <u>continuous</u> extension of φ requires adding an image pt for each tangent direction.