Kahler differentials

Friday, November 29, 2024 7:34 AM

Start of 05.11.2025

The goal is to develop a Theory of cotangent sheaf of a scheme.

Terminology: • Let A be a commutative ring. An A-alg is a ring R with a ring homomorphism $\varphi: A \to R$.

- In This case R becomes an A module: For $a \in A$, $r \in R$, $ar = \varphi(a) r$
- R, R' A-algbs; a ring homomorphism $f: R \to R'$ is an A-alg map if $f(ar) = a f(r)$ $\forall a \in A$, $r \in R$.

Rmk: In practice we never mention The ring homomorphism $A \to R$ imposing the A alg structure.

Derivations are suitable adaptations of The notion of derivatives

Def: Let R be an A-algebra and M be an R mod. An A-linear derivation from R to M is a map $d: R \to M$ s.t

① d is A-linear: i.e $d(\alpha + \beta) = d(\alpha) + d(\beta)$ and $d(\lambda \alpha) = \lambda d(\alpha)$ $\forall \alpha, \beta \in R$, $\lambda \in A$,

② $d(\alpha\beta) = \alpha d\beta + \beta d\alpha$. (Leibniz rule).

The A-mod of $\forall$ A-linear der $A \to R$ is denoted $\mathrm{Der}_A(R, M)$

- Let $(T, \mathcal{O}_T)$ be a ringed space, $\tilde{\mathcal{O}}_T$ be a sheaf of $\mathcal{O}_T$-algbs, $\mathcal{F}$ be an $\tilde{\mathcal{O}}_T$-mod. An $\mathcal{O}_T$-linear map $d: \tilde{\mathcal{O}}_T \to \mathcal{F}$ is called a derivation if for every open $U \subseteq T$ $d_U: \tilde{\mathcal{O}}_T(U) \to \mathcal{F}(U)$ is an $\mathcal{O}_T(U)$-lin derivation.

Rmk. $d: R \to M$ be A-lin derivation. Then $d(\varphi(a)) = 0$ $\forall a \in A$. Indeed $d(\varphi(a)) = \varphi(a) d(1)$. and $d(1) = d(1 \cdot 1) = 1 \cdot d(1) + 1 \cdot d(1) \Rightarrow d(1) = 0$

Eg: • $k[x] \to k[x]$, k field.
$f \longmapsto \frac{d}{dx} f$ is a k-lin derivation

• $k[x] \longrightarrow k[x] \xrightarrow{\frac{d}{dx}} [x]$ $\qquad d(f) = \frac{d}{dx}(f(x^2))$
$x \longmapsto x^2$ ring homo $\qquad = 2x \cdot \frac{df}{dx}(x^2)$
is a derivation.

Thm/Def Given $\varphi: A \to R$ as above, There is an R-mod denoted $\Omega_{R/A}$ and an A-linear derivation $d_{R/A}: R \to \Omega_{R/A}$ such That any A-lin derivation $d: R \to M$ factors uniquely as

$d_{R/A}$

such "that any A-lin derivation $d: R \to M$ factors" uniquely as

$$\begin{tikzcd} R \arrow[r, "d_{R/A}"] \arrow[dr, "d"'] & \Omega_{R/A} \arrow[d, dashed, "\exists!\, \varphi_d"] \\ & M \end{tikzcd}$$

where $\varphi_d: \Omega_{R/A} \to M$ is R-linear.

- The pair $(\Omega_{R/A}, d_{R/A})$ is unique upto a unique isom.
- $\Omega_{R/A} :=$ module of Kahler differentials of R over A

  $d_{R/A} :=$ the universal derivation of the A-alg R.

Pf. • uniqueness

Given another pair $(\tilde{\Omega}_{R/A}, \tilde{d}_{R/A})$ get a commutative diag

$$\begin{tikzcd} & R \arrow[dl, "d_{R/A}"'] \arrow[dr, "\tilde{d}_{R/A}"] & \\ \Omega_{R/A} \arrow[rr, shift left, "\varphi_{d_{R/A}}"] & & \tilde{\Omega}_{R/A} \arrow[ll, dashed, shift left, "\tilde{\varphi}_{\tilde{d}_{R/A}}"] \end{tikzcd}$$

Now note $d_{R/A} = \tilde{\varphi} \cdot \varphi \cdot d_{R/A}$

By the uniqueness of the factorization $\tilde{\varphi} \cdot \varphi = id$.

Similarly $\varphi \cdot \tilde{\varphi} = id$.

- Existence: (i) Consider the set of symbols $\{\bar{d}(r)\}_{r \in R}$

  Let $\Omega' :=$ free R mod with basis $\{\bar{d}(r)\}_{r \in R}$.

  set $N =$ R-sub mod spanned by

  $\{\bar{d}(a) \mid a \in A\} \cup \{\bar{d}(r_1 + r_2) - \bar{d}(r_1) - \bar{d}(r_2) \mid r_1, r_2 \in R\}$

  $\cup \{\bar{d}(r_1 r_2) = r_1 dr_2 + r_2 dr_1\}$

  Set $\Omega_{R/A} = \dfrac{\Omega'}{N}$, define $d_{R/A}: R \to \Omega_{R/A}$

  by $d_{R/A}(r) =$ image of $\bar{d}(r)$ in $\bar{\Omega}/N$.

  - $d_{R/A}$ is an A-lin derivation.

- Universal property: Given a derivation $d: R \to M$

  Define $\bar{\varphi}: \bar{\Omega} \to M$ by $\bar{\varphi}(\bar{d}r) = d(r)$

  $\because$ d derivation $d(N) = 0$, so get a unique map

  $\varphi_d: \bar{\Omega}/N = \Omega_{R/A} \to M$ st

  $\varphi_d(\underbrace{dr \,(\mathrm{mod}\, N)}_{= d_{R/A}(r)}) = d(r)$.

Example. $R = A[x_1, \dots, x_n]$ polynomial ring.

$\Omega_{R/A} \underset{\text{free mod}}{=} R\,dx_1 \oplus \dots \oplus R\,dx_n$

$d_{R/A}: R \longrightarrow \Omega_{R/A}$ is given by

$$d_{R/A}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i$$

<u>Claim:</u> Given a derivation $d: R \to M$

$$d(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, d(x_i)$$

· Since both the sides are A-linear enough to check for ~~[illegible]~~ $f = x_1^{a_1} \cdots x_n^{a_n}$.

So d factorizes as

$$\begin{tikzcd} R \arrow[r, "d_{R/A}"] \arrow[dr, "d"'] & \bigoplus R\, dx_i = \Omega_{R/A} \arrow[d, dashed] \\ & M \end{tikzcd} \qquad dx_i \mapsto d(x_i)$$

<u>Proof.</u> Let I be the kernel $R \otimes_A R \longrightarrow R$
$$x \otimes y \longmapsto xy$$

Consider $d_0: R \longrightarrow I/I^2$ given by $d_0(r) =$ image of $1 \otimes r - r \otimes 1 \in I/I^2$

Then $(I/I^2, d_0) =$ (module of Kähler differentials, universal derivation).

<u>Pf.</u> Note $R \otimes_A R \longrightarrow R$ induces an isom

$$R \otimes_A R / I \xrightarrow{\sim} R$$

Since $I \cdot I/I^2 = 0$, I/I^2 is naturally an R-mod:

For $r \in R$, $y \in I/I^2$

$$r \cdot y := (1 \otimes r) \cdot y = (r \otimes 1) \cdot y$$

· Check $d_0: R \to I/I^2$ is a derivation.

· Now given an A-lin derivation $d: R \to M$

extend d uniquely to an R-lin map

$$\tilde{d}: R \otimes_A R \longrightarrow M$$
$$\tilde{d}(r_1 \otimes r_2) = r_1\, d(r_2)$$

We show $\tilde{d}(I^2) = 0$.

<u>Claim:</u> Consider $R \otimes_A R$ as an R-mod via the map

$R \longrightarrow R \otimes_A R$
$r \longmapsto r \otimes 1$ (i.e. the left action)

Then I is gen by $\{1 \otimes r - r \otimes 1 \mid r \in R\}$ as an R-mod.

<u>Pf.</u> Take $\sum x_i \otimes y_i \in I \Rightarrow \sum x_i y_i = 0$

$$\sum x_i \otimes y_i = \sum \left[x_i(1 \otimes y_i - y_i \otimes 1) + x_i y_i \otimes 1\right]$$
$$= \sum x_i (1 \otimes y_i - y_i \otimes 1) + \underbrace{\left(\sum x_i y_i\right)}_{0} \otimes 1$$

$$\sum x_i \otimes y_i - \sum [x_i (1 \otimes y_i \ \dots$$

$$= \sum x_i (1 \otimes y_i - y_i \otimes 1) + \underbrace{\left(\sum x_i y_i\right)}_{=0} \otimes 1 \qquad \text{D}.$$

$\tilde{d}(I^2) = 0$ follows once we show

$$\tilde{d}\left((1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1) \right) = 0$$

$$\iff \tilde{d}(1 \otimes xy - y \otimes x - x \otimes y + xy \otimes 1) = 0$$

$$\iff d(xy) - y\,dx - x\,dy + xy\,d(1) = 0$$

$$\iff d(xy) = y\,dx + x\,dy$$

Note by def $\tilde{d}(d_0(r)) = \tilde{d}(1 \otimes r - r \otimes 1)$

$$= 1\,dr - r\,d(1)$$

$$= dr$$

So we have a commutative diag

$$\begin{tikzcd}
 & I/I^2 \arrow[dr, "\tilde{d}|_{I/I^2}"] & \\
R \arrow[ur, "d_0"] \arrow[rr, "d"'] & & M
\end{tikzcd}$$

The uniqueness of the factorization follows from the claim.

<u>Thm.</u> 1) Given an A-alg map $R \xrightarrow[\psi]{} R'$, there exists <u>a unique map</u>

$$d\psi : \Omega_{R/A} \to \Omega_{R'/A} \text{ such that}$$

$$d\psi(d_{R/A}(r)) = d_{R'/A}(\psi(r)).$$

2) For a multiplicatively closed subset $S \subseteq R$
The canonical map $R \xrightarrow{i} S^{-1}R$ induces an isom

$$S^{-1}(\Omega_{R/A}) \xrightarrow[S^{-1}(di)]{\sim} \Omega_{S^{-1}R/A}.$$

So in particular, for a prime ideal p of R.

$$(\Omega_{R/A})_p \xrightarrow{\sim} \Omega_{R_p/A} \xrightarrow[\text{H.W}]{\sim} \Omega_{R_p/A_q}$$

Use 3,

where q is the inverse image of p under $A \to R$.

3) Let A be an A_0-alg. The maps $A_0 \xrightarrow[\psi]{} A \xrightarrow[\varphi]{} R$ <u>induces</u> an <u>exact seq</u>

$$\Omega_{A/A_0} \otimes_A R \longrightarrow \Omega_{R/A_0} \longrightarrow \Omega_{R/A} \longrightarrow 0$$

$$d_{R/A_0}(r) \longmapsto d_{R/A}(r)$$

$$d_{A/A_0}(a) \otimes r \longmapsto r \cdot d_{R/A_0}(\varphi(a)).$$

4) Given an ideal $J \subseteq R$, there is an exact seq

$$\dots \longrightarrow 0 \ : \text{Conormal}$$

4) Given an ideal $J \subseteq R$, There is an exact seq

$$J/J^2 \longrightarrow \Omega_{R/A} \otimes_R R/J \longrightarrow \Omega_{R/J/A} \longrightarrow 0$$: Conormal sequence

$$d_{R/A}(r) \otimes (r' \ (\mathrm{mod}\ J)) \longmapsto r' d_{R/J/A}(r \ (\mathrm{mod}\ J))$$

$$\alpha \ (\mathrm{mod}\ J^2) \longmapsto d_{R/A}(\alpha) \otimes 1$$

5) Let B be an A-alg. The natural map

$$\Omega_{R/A} \otimes_A B \longrightarrow \Omega_{R \otimes_A B/B} \text{ is an isom of } R \otimes_A B$$

mods

$$d_{R/A}(r) \otimes b \longmapsto b \, d_{R\otimes_A B/B}(r \otimes 1) = d_{R \otimes_A B/B}(r \otimes b)$$

Pf. In (3), (4), The idea is to apply $\mathrm{Hom}_R(-, M)$ and check The resulting sequences are left exact, for $\forall M \in \mathrm{Mod}_R$ (for 3), $M \in \mathrm{Mod}_{R/J}$ (for 4) We first check (4)

Note for $\alpha, \beta \in J$,

$$d_{R/A}(\alpha\beta) = \alpha \, d_{R/A}(\beta) + \beta \, d_{R/A}(\alpha)$$
$$\in J \Omega_{R/A}$$

So $d_{R/A}(\alpha\beta) = 0 \in \Omega_{R/A} \otimes_R R/J$.

Thus The map $J \longrightarrow \Omega_{R/A} \otimes_R R/J$ sends J^2 to zero.

For $M \in \mathrm{Mod}_{R/J}$ The resulting seq is

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(J/J^2, M) & \longleftarrow & \mathrm{Hom}_R(\Omega_{R/A} \otimes_R R/J, M) & \longleftarrow & \mathrm{Hom}_{R/J}(\Omega_{R/J/A}, M) & \longleftarrow & 0 \\
\| & & \| & & \| & & \\
\mathrm{Hom}_R(J/J^2, M) & \longleftarrow & \mathrm{Der}_A(R, M) & \longleftarrow & \mathrm{Der}_A(R/J, M) & \longleftarrow & 0
\end{array}$$

Since $R \to R/J$ is surj, The injectivity is clear.
To check The exactness in The middle, take $d \in \mathrm{Der}_A(R, M)$ s.t $d(J) = 0$.

Then The induced A-lin map $\bar{d} : R/J \to M$ is also a derivation, and d is The image of $\bar{d}$. (End of 05.11.2025)

e.g: $R = \dfrac{\mathbb{C}[x,y]}{(x^3+y^2)}$,

$$\frac{(x^3+y^2)}{(x^3+y^2)^2} \xrightarrow{d} \Omega_{\mathbb{C}[x,y]/\mathbb{C}} \otimes \frac{\mathbb{C}[x,y]}{x^3+y^2} \longrightarrow \Omega_{R/\mathbb{C}} \longrightarrow 0$$

$$\text{So } \Omega_{R/\mathbb{C}} \cong \frac{\frac{\mathbb{C}[x,y]}{(x^3+y^2)} dx \oplus \frac{\mathbb{C}[x,y]}{(x^3+y^2)} dy}{3x^2 dx + 2y\, dy}$$

Def: R is called a <u>finite type A algebra</u> if $\exists$ finitely many elements $x_1, x_2, \ldots, x_n \in R$ s.t any elt of R is a polynomial

Def: R is called a finite type A algebra if $\exists$ finitely many elements $x_1, x_2, \ldots, x_n \in R$ s.t any elt of R is a polynomial in $x_1, \ldots, x_n$ with coefficient in (the image of) A.

Rmk: This means the A-alg map from the polynomial ring
$$A[Y_1, \ldots, Y_n] \longrightarrow R \text{ is surjective}$$
$$Y_i \longmapsto x_i$$
Let I be the kernel. So $R \xrightarrow{\sim} A[Y_1 \ldots Y_n]/I$.

H.W. R is a finite type (f.t) A alg iff $R \xrightarrow{\sim} \dfrac{A[Y_1 \ldots Y_n]}{I}$ for some n, and I.

Prop. R f.t A-alg. Then $\Omega_{R/A}$ is a finitely generated R module

Pf. Use conormal seq.

Globalize.

$X \to S$ be a morphism of schemes. $\Delta: X \to X \times_S X$ be the diagonal map.

Claim: $\exists$ an open $U \subseteq X \times_S X$ s.t Δ factors as $X \xrightarrow{\Delta_U} U \hookrightarrow X \times_S X$, where Δ' is a closed immersion.

Pf. Choose an affine open covering $S = \cup S_i$, $X = \cup V_i$ where V_i maps to S_i.

Take $U = \bigcup_i V_i \times_S V_i = \bigcup V_i \times_{S_i} V_i$.

$\Delta_U^{-1}(V_i \times_{S_i} V_i) = V_i$. Thus, we have a factorization of Δ
$$X \xrightarrow{\Delta_U} U \to X \times_S X$$
$(\Delta_U^{-1})(V_i \times_S V_i) = V_i$, and $\Delta_{U|V_i}: V_i \to V_i \times_{S_i} V_i$ is the diagonal map. As V_i is affine $\Delta'_{|V_i}$ is a closed immersion $\forall i$. So Δ_U is a closed immersion.

Def Let $X \to S$ be a morphism of schemes. Consider the diagonal map $\Delta: X \to X \times_S X$.

Choose an open $U \subseteq X \times_S X$ s.t we have a factorization $X \xrightarrow[\text{closed}]{\Delta_U} U \hookrightarrow X \times_S X$

Let $\mathcal{J}_U = \operatorname{Ker}(\mathcal{O}_U \to \Delta_{U*}\mathcal{O}_X)$

Define $\Omega_{X/S} = \Delta^*(\mathcal{J}_U/\mathcal{J}_U^2)$ - This $\mathcal{O}_X$-mod sheaf of Kähler differentials of the S-scheme X.

Define $\Omega_{X/S} = \Delta^*(\mathcal{J}_U/\mathcal{J}_U^2)$ - This $\mathcal{O}_X$-mod sheaf of Kähler differentials of the S-scheme X.

Rmk. $\Omega_{X/S}$ indeed does not depend on the choice of U.

For two choices $U', U \subseteq X \times_S X$, one can also take the factorization $X \xrightarrow[\Delta_{U\cap U'}]{} U \cap U' \hookrightarrow X \times_S X$. Using the diagram $X \hookrightarrow U \cap U' \hookrightarrow U$, conclude $\mathcal{J}_{U\cap U'} = \mathcal{J}_U|_{U \cap U'}$ and $\Delta_U^*(\mathcal{J}_U/\mathcal{J}_U^2) = \Delta_{U\cap U'}^*(\mathcal{J}_{U\cap U'}/\mathcal{J}_{U\cap U'}^2)$

Prop. There is a $f^{-1}(\mathcal{O}_S)$ linear derivation $d_{X/S}: \mathcal{O}_X \to \Omega_{X/S}$ such that for any $f^{-1}(\mathcal{O}_S)$ linear derivation d factors uniquely through $d_{X/S}$

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "d"] \arrow[dr, "d_{X/S}"'] & \mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X} \\ & \Omega_{X/S} \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

So $(\Omega_{X/S}, d_{X/S})$ is the unique (upto isom) with this property.

Prop. $\Omega_{X/S}$ is a quasi-coherent sheaf.

Pf. Fix an $U \underset{\text{open}}{\subseteq} X \times_S X$ s.t we have factorization

$$X \xrightarrow[\text{closed}]{\Delta_U} U \xrightarrow[\text{open}]{} X \times_S X$$

Then $\mathcal{J}_U$ is q.coh since $\mathcal{J}_U = \mathrm{Ker}(\mathcal{O}_U \to (\Delta_U)_*\mathcal{O}_X)$ and Δ_U is a closed immersion. So $(\Delta_U)_*\mathcal{O}_X$ is q.coh. $\square$

So $\mathcal{J}_U/\mathcal{J}_U^2$ is Qcoh and $\Delta_U^*(\mathcal{J}_U/\mathcal{J}_U^2)$ is Qcoh.

Rmk: Choose affine opens $U \subseteq X$, $V \subseteq S$, s.t U maps to V.

By our construction $\Omega_{U/V}$ is qcoh with $\Omega_{U/V}(U) = \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$

Again by qcoh of $\Omega_{X/S}$, $\Omega_{X/S}|_U \cong \widetilde{\Omega}_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$

So $\Omega_{X/S}$ is obtained by gluing $\widetilde{\Omega}_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$ and $d_{X/S}$ is obtained by gluing the corresponding local universal derivations.

Prop. $\varphi: X \to S$ be the morphism inducing the S-scheme structure on X. For $x \in X$, $(\Omega^1_{X/S})_x \cong \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{S,\varphi(x)}}$

Pf. W.L.O.G both X, S are affine

So $(\Omega^1_{X/S})_x \cong (\Omega^1_{\mathcal{O}_X(X)/\mathcal{O}_S(S)})_x$

Pf. W.L.O.G both X, S are affine
so $(\Omega^1_{X/S})_x \cong (\Omega^1_{\mathcal{O}_X(X)/\mathcal{O}_S(S)})_x$
Use results from Kähler differentials and localization.

Prop. Let $X \to S$ be a finite type morphism, S is noetherian. Then $\Omega_{X/S}$ is a coherent sheaf.

Proof follows from the more general observation. and that $\Omega_{X/S}$ is quasi-coherent.

Prop. Let A be a commutative ring. R be a finite type A-alg. Then $\Omega_{R/A}$ is a finitely gen R-mod.

Pf. Write $R = A[x_1 \dots x_n]/I$.
The conormal seq gives a surj

$$\Omega_{A[x_1 \dots, x_n]/I\,/A} \otimes_A R \twoheadrightarrow \Omega_{R/A}$$
$$\cong \bigoplus_{i=1}^n R\,dx_i$$

Regularity vs Kähler differentials

Terminology: $\mathcal{F}$, $\mathcal{O}_X$ module where X is a scheme. The fibre of $\mathcal{F}$ at $x \in X$ is $\mathcal{F}_x/m_x\mathcal{F}_x$, where m_x is the maximal ideal of $\mathcal{O}_{X,x}$.

Note $\mathcal{F}_x/m_x\mathcal{F}_x$ is a $k(x) := \mathcal{O}_{X,x}/m_x$ vector space.

Prop. Let (R, m) a noetherian local ring containing a field k such that the composition
$$k \to R \to R/m \text{ is an isomorphism}$$
Then the natural map $m/m^2 \to \Omega_{R/k} \otimes_R R/m$ is an isom.

Pf. The conormal seq gives an exact seq
$$m/m^2 \longrightarrow \Omega_{R/k} \otimes_R R/m \to \Omega_{R/m\,/k} \to 0$$
Since $\Omega_{R/m/k} = 0$, we have a surjective map
$$m/m^2 \longrightarrow \Omega_{R/k} \otimes_R R/m$$
We claim that this map is also injective. For that take an $R/m = k$ mod M. We check that the induced map
$$\mathrm{Der}_k(R, M) = \mathrm{Hom}_k(\Omega_{R/k} \otimes_R R/m, M) \longrightarrow \mathrm{Hom}_k(m/m^2, M)$$
is surj. • Here M is thought of as an R-mod by

$\ldots \operatorname{Hom}_k(\ldots /_k \otimes_R \ldots /m, \ldots) \to \operatorname{Hom}_R(\ldots)$

is surj. • Here M is thought of as an R-mod by $R \to R/m$.

Fix a k-lin map $\varphi : m/m^2 \to M$.

Note that the inclusion map $k \to R$ gives a k-linear section (a right inverse) of the projection. Thus as a k-vector space $R = k \oplus m$ and $R/m^2 = k \oplus m/m^2$.

Define $d\varphi : R/m^2 \to M$ by setting $d\varphi|_k = 0$ and $d\varphi(m/m^2) = \varphi$

Claim: $d\varphi$ is a k-linear derivation.

Pf k-linearity is clear. We check that $d\varphi$ satisfy the Leibniz rule.

For $x, y \in R$, write $\bar{x} = \lambda_x + \alpha_x \in R/m^2$

$\bar{y} = \lambda_y + \alpha_y \in R/m^2$

where $\lambda_x, \lambda_y \in k$, $\alpha_x, \alpha_y \in m/m^2$

Then $\bar{x} \cdot \bar{y} = \lambda_x \alpha_y + \lambda_y \alpha_x + \lambda_x \lambda_y$

So $d\varphi(\bar{x}\cdot\bar{y}) = \lambda_x \, d\varphi(\alpha_y) + \lambda_y \, d\varphi(\alpha_x)$ $[\because d\varphi|_k = 0]$

$= (\lambda_x + \alpha_x)\, d\varphi(\lambda_y + \alpha_y)$

$+ (\lambda_y + \alpha_y)\, d\varphi(\lambda_x + \alpha_x)$ $[\because d\varphi|_k = 0,\ m \cdot M = 0]$

$= \bar{x}\, d\varphi(\bar{y}) + \bar{y} \cdot d\varphi(\bar{x})$ ☒

Since $d\varphi$ maps to φ, we get the surjectivity.

Prop: X be an k-scheme, $x \in X$ be a k-rational point (i.e. there is a k-scheme map $\operatorname{Spec}(k) \to X$ with image x)

Then, the fibre of $\Omega^1_{X/k}$ at x is isom to m_x/m_x^2

Pf. $(\Omega^1_{X/k})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/m_x \cong \Omega^1_{\mathcal{O}_{X,x}/k} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/m_x$

As x is a k-rat point $k \hookrightarrow \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/m_x$ is an isom.

Use the last result.

Prop. Let $L \supseteq k$ be a finite field extension.

$\Omega_{L/k} = 0 \iff L \supseteq k$ is separable. (HW)

Def: A field extension $K \subseteq L$ is called finitely generated if $\exists\ x_1, \ldots, x_n \in L$ s.t any elt of L can be written as $f(x_1, \ldots, x_n)/g(x_1, \ldots, x_n)$ where f, g are some polynomials with coef in K.

Equivalently, L is the quotient field of a f.t K algebra which is a domain.

$\ldots\ \mathbb{C}(X)[Y]$

is a domain.

Eg: a) $\mathbb{C} \subseteq \dfrac{\mathbb{C}(X)[Y]}{(Y^2 - X^3)}$

b) X/k f.t integral scheme; η be the generic pt of X.

Then $k(X) := \mathcal{O}_{X,\eta}$ is a finitely gen field extension of k.

Thm: Let K be a perfect field (i.e $ch(k)=0$ or $ch(k)=p$, $K=K^p$)
$K \subseteq L$ be a finitely generated field extension;
n = transcendence degree of $K \subseteq L$.
Then $\exists$ a transcendence basis $x_1, \dots, x_n$ of $K \subseteq L$ s.t
$K(x_1, \dots x_n) \subseteq L$ is finite and separable.

Prop: Let k be an algebraically closed field.
Then $\dim_L \Omega_{L/k} = \mathrm{trdeg}_k L$ (in fact equal)

Pf Let $d = \mathrm{trdeg}_k L$. Since k is algebraically closed, we can find a transcendence basis L/k, $x_1, \dots, x_d \in L$ such That

$$k(x_1 \dots x_d) \subseteq L \text{ (finite) separable.}$$

We have an exact seq

$$\Omega_{k(x_1 \dots x_d)/k} \otimes L \longrightarrow \Omega_{L/k} \longrightarrow \Omega_{L/k(x_1 \dots x_d)} \longrightarrow 0$$

Since $\Omega_{L/k(x_1 \dots x_d)} = 0$, and $\Omega_{k(x_1 \dots x_d)/k} \otimes_{k(x_1 \dots x_d)} L \cong L^{\oplus d}$

Chose a f.t k-alg A s.t $\mathrm{Frac}(A) = L$.
Then $\Omega_{L/k} \xrightarrow{\sim} \Omega_{A/k} \otimes_A \mathrm{Frac}(A)$. Since $\Omega_{A/k}$ is f.g $/A$,
$\exists$ $0 \neq f \in A$ such that $\Omega_{A/k} \otimes_A A[1/f] \simeq \Omega_{A_f/k}$ is free.

Take a maximal ideal m of A_f. Note $k \to A_m \to A_m$ is an isom as k is alg closed.

Thus $m/m^2 \simeq (\Omega_{A_f/k})_m \otimes_A A/m$

$$\Rightarrow \mathrm{rk}_{A_f} \Omega_{A_f/k} = \dim_{A/m} (\Omega_{A_f/k})_m \otimes A_m = \mathrm{rk}_{A/m} m/m^2$$

But $\mathrm{rk}_{A/m} m/m^2 \geq \dim A_m = \dim A = \mathrm{tr}_k L$

So $\mathrm{rk}_L \Omega_{L/k} = \mathrm{rk}_{A_f} \Omega_{A_f/k} = \dim_k L$.

Thm: Let k be an algebraically closed field.
X/k be a finite type integral scheme. Then X is regular at x (i.e $\mathcal{O}_{X,x}$ is a regular ring) iff

X/k be a finite type integral scheme. Then X is regular at x (i.e. $\mathcal{O}_{X,x}$ is a regular ring) iff $(\Omega_{X/k})_x$ is a free $\mathcal{O}_{X,x}$ module

Pf. Since the involved statements are local at x, W.L.O.G we can assume $X = \mathrm{Spec}(R)$, where R is a finite type k-alg, which is a domain. and x corresponds to the prime ideal $p \subseteq R$

Consider the conormal sequence

$$\frac{pR_p}{p^2R_p} \longrightarrow \Omega_{R_p/k} \otimes_{R_p} \frac{R_p}{pR_p} \longrightarrow \Omega_{k(p)/k} \to 0 \quad \text{where } k(p) = \frac{R_p}{pR_p} = \mathrm{Frac}(R/p).$$

Assume R_p is regular. Then $\dim_{k(p)} \frac{pR_p}{p^2R_p} = \dim R_p$

By the Lemma above, $\dim_{k(p)} \Omega_{k(p)/k} = \dim R/p$
$= \mathrm{tr\,deg}_k(k(p))$

So $\dim_{k(p)} \Omega_{R_p/k} \otimes k(p) \leq \dim R/p + \dim R_p$
$= \dim(X)$ [$\because R$ is f.t/k and R is a domain]

We claim that the above inequality is in fact an equality. Indeed, the L.H.S is the min # of gens of the R_p-mod $\Omega_{R_p/k}$. In case of strict inequality $\Omega_{R_p/k}$ is gen /R_p by $x_1, x_2, \dots, x_n$ where $n < \dim X$.

But $\Omega_{\mathrm{Frac}(R)/k} = \Omega_{R_p/k} \otimes_{R_p} \mathrm{Frac}(R_p)$ is then gen by the image of $x_1, \dots, x_n$ in the localization. This contradicts the Prop above.

The freeness of $\Omega_{R_p/k}$ now follows from the lemma below.

Lemma: Let (A, m) be a noetherian domain, M be a finitely generated A-module such that
$$\dim_{A/m} M/mM = \dim_{\mathrm{Frac}(A)} M \otimes_A \mathrm{Frac}(A)$$

Then M is a free module

Pf. Let $\mu = \dim_{A/m} M/mM$.

Consider an exact seq $0 \to K \to A^{\oplus\mu} \to M \to 0$ where K is the kernel where the map $A^{\oplus\mu} \to M$ comes from a choice of min set of generators of M.

Applying $\otimes_A \mathrm{Frac}(A)$ to the above exact seq, get $K \otimes_A \mathrm{Frac}(A) = 0$.

But A is a domain $\Rightarrow$ K is torsion free

So $K \otimes_A \mathrm{Frac}(A) = 0 \Rightarrow K = 0$. $\square$

- Now assume $\Omega_{R_p/k} \simeq (\Omega_{R/k})_p$ is free.

Then there is $f \notin p$ s.t $\Omega_{R/k} \otimes_R R[1/f]$ is locally free –

Choose a maximal ideal $m \in R$ of $R[1/f]$ that contains

Then There is $\exists f \notin p$ s.t $\Omega_{R/k} \otimes_R R[1/f]$ is locally free.

Choose a maximal ideal $m \in R$ of $R[1/f]$ that contains p. $\Omega_{R_m/k}$ is locally free of rk = dim X = dim R_m

But $\Omega_{R_m/k} \otimes_{R_m} \underset{\underset{k, \text{ as } k \text{ is alg closed}}{\shortparallel}}{\frac{R_m}{mR_m}} \xrightarrow{\sim} m/m^2$

So $\dim R_m = \mathrm{rk}_{R_m}(\Omega_{R_m/k}) = \dim m/m^2$

$\Rightarrow$ R_m is regular

Now we use the (non-trivial) fact that localization of a regular local ring regular to conclude $R_p \cong (R_m)_{pR_m}$ is regular.

Prop. Let X be a finite type integral scheme over an algebraically closed field

Then Regular locus of X $:= \{x \in X \mid X \text{ is reg at } x, \text{ i.e } \mathcal{O}_{X,x} \text{ is reg}\}$

$= \{x \in X \mid (\Omega_{X/k})_x \text{ is free}\}$

In particular 1) The regular locus of X is open in X

2) X is regular $\Leftrightarrow$ $\Omega_{X/k}$ is a locally free module of rk = dim X.