Algebraic Geometry

Lecture Notes

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1 Lecture I

We begin by recalling the theory of affine algebraic varieties, which we glimpsed during last year's lectures and exercises. This is useful both to build intuition and motivation for the language of schemes, and also to remember some fundamental results from commutative algebra. I advise you to look into the numerous references (the most important are listed on the Moodle page). The aim of this first lecture is also to define some of the main objects which we shall later study via the language of schemes.

1.1 Affine algebraic varieties

Let K be an algebraically closed field, which we fix from now on. The weak Nullstellensatz says that we can identify K^n with the set of maximal ideals of $K[x_1, \dots, x_n]$, and it clearly expresses the first fundamental bridge between algebra and geometry. The full Nullstellensatz instead says that for every ideal $I \subset K[x_1, \dots, x_n]$ we have

$$I(V(I)) = \sqrt{I}.$$

(Why does this imply the weak form?)

Recall that we can endow K^n with the Zariski topology, where the closed subsets are of the form $V(I)$ for $I \subset K[x_1, \dots, x_n]$. We also call the subsets $V(I)$ the *algebraic subsets* of K^n. These are very explicitly defined: since $K[x_1, \cdots, x_n]$ is Noetherian, every ideal is finitely generated. Write $I = (f_1, \dots, f_d)$, then

$$V(I) = \{(x_i) \in K^n : \ f_1(x_1, \dots, x_n) = \cdots = f_d(x_1, \dots, x_n) = 0\},$$

which in turn corresponds to maximal ideals $\mathfrak{m} \subset K[x_1, \dots, x_n]$ with $I \subset \mathfrak{m}$.

These subsets are the basic building blocks of algebraic geometry, and we shall spend some time understanding their properties. First of all, we can endow $V(I)$ with a topology in two equivalent ways (verify this): either restrict the Zariski topology of K^n to $V(I)$, or identify $V(I)$ with the set of maximal ideals $\mathfrak{m} \subset A := K[x_1, \dots, x_n]/I$ and endow it with the topology generated by

$$V(J) := \{\mathfrak{m} \subset A : \ J \subset \mathfrak{m}\}$$

for ideals $J \subset A$.

The topology of $V(I)$ is very different from what we are normally used to, and enjoys many finiteness properties:

Definition 1.1. A topological space X is said to be *Noetherian* if any descending chain of closed subsets

$$Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$$

eventually stabilizes.

The fact that $K[x_1, \cdots, x_n]$ is a Noetherian ring then readily implies that any $V(I)$ is also Noetherian. Recall that we defined the topological space $\mathrm{Spec}(A)$ for any commutative ring. Show that if A is Noetherian then $\mathrm{Spec}(A)$ is Noetherian (is this an if and only if?). Note, on the other hand, that the usual topology of $\mathbb{R}^n$ is never Noetherian for $n \geq 1$.

Theorem 1.2 (Noetherian induction). *Let X be a Noetherian topological space and P be a property of the closed subsets of X. Assume $P(\emptyset)$ is true, and that for every closed subset $Z \subset X$, the fact that $P(W)$ is true for every proper closed $W \subsetneq Z$ implies that also $P(Z)$ is true. Then P is true for all closed subsets of X.*

This or similar formulations appear many times in proofs, for example when studying coherent cohomology. We use it now to show that every $V(I)$ can be decomposed uniquely into a union of irreducible subsets:

Definition 1.3. A Noetherian topological space X is *irreducible* if for every two closed subsets $X_1, X_2 \subsetneq X$ we have $X_1 \cup X_2 \subsetneq X$.

Recall that $V(I)$ is irreducible if and only if $\sqrt{I}$ is prime.

Theorem 1.4. *Every closed subset $V(I) \subset K^n$ can be decomposed uniquely as a finite union of distinct irreducible closed subsets.*

Proof. Prove it using Noetherian induction. Compare it to the proof that every radical ideal of $K[x_1, \dots, x_n]$ decomposes uniquely into a product of prime ideals. We omit the details. □

The subsets $V(I)$ with I prime are called *affine algebraic varieties.*

Remark 1.5. To gain geometric intuition for algebraic varieties, assume that $\mathrm{char}(K) = 0$ and that $|K| \leq |\mathbb{C}|$ (this assumption is very reasonable). Then one can embed $K \subset \mathbb{C}$ and consider $V(I)$ as an algebraic subset of $\mathbb{C}^n$. But this is also closed for the usual topology, and we can study its properties with tools from topology, real analysis, and complex analysis. As one will see, many topological properties of $V(I)$ can be obtained by purely algebraic methods without leaving the language of algebraic geometry.

Many times we can also embed $K \subset \mathbb{R}$, and $V(I)$ yields a closed subset of $\mathbb{R}^n$. Although this can still be used to get geometric insights into $V(I)$, we remark that $V(I)$ may behave strangely (e.g. it may be empty). This is why classical algebraic geometry is formulated over algebraically closed fields: so that we see all points of $V(I)$ over K.

The topology of irreducible Noetherian topological spaces is very coarse:

Proposition 1.6. *If X is an irreducible topological space and $U \subset X$ is a non-empty open subset, then U is dense.*

Proof. Assume that $\overline{U} \neq X$, so $V = X \setminus \overline{U}$ is open and non-empty. Note that $V \cap U = \emptyset$ and that $X = \overline{U} \cup \overline{V}$, contradicting irreducibility. □

Thus affine algebraic varieties are never Hausdorff (T_2) in the classical sense. In fact, the right concept is the one of *separatedness*, as we shall see later in the course.

Observation 1.7. *If we wish to have a faithful dictionary between algebra and geometry, we already see that something is amiss: different ideals can yield the same algebraic subset. Thus the algebraic side carries more information than the geometric side in classical algebraic geometry. This might seem like a minor inconvenience at first, but the deeper one digs into the structure of algebraic varieties, the more essential it becomes. The interpretation of this phenomenon in* *terms of nilpotents in the structure sheaf of algebraic varieties—and in general the realization of the key role played by nilpotent elements in algebraic geometry—is one of the leading motivations for the abstract language of schemes.*

1.1.1 Examples

At this stage, we cannot say much more, but let us list some basic examples:

1. **Linear varieties.** These are the zero sets of degree one polynomials. They can always be written as $P + W$, where $P \in K^n$ is a point and $W \subset K^n$ is a vector subspace.

2. **Conics in $\mathbb{A}^2_K$.** Assume that $\mathrm{char}(K) \neq 2$ and let $Q(x, y) \in K[x, y]$ be a quadratic polynomial. Then $C = V(Q) \subset K^2$ is called an affine conic. When is C irreducible? If C is not irreducible, in which ways can it decompose?

3. **The Fermat curve.** For $n \geq 1$ let
$$F_n = V(x^n + y^n - 1) \subset K^2.$$
Show that F_n is always irreducible (what happens when $\mathrm{char}(K)$ divides n?). Fermat's Last Theorem then says that the only $(a, b) \in \mathbb{Q}^2$ satisfying $a^n + b^n = 1$ are $(1, 0)$ and $(0, 1)$. In fact, much of algebraic number theory boils down to the study of polynomial equations over non-closed fields. Since modern algebraic geometry works with any scheme as a base (not necessarily a field), its language automatically incorporates many classical concepts of algebraic number theory (e.g. ideal class groups, ramification theory, etc.). It will eventually allow one to show how the topological properties of a curve $C \subset \mathbb{C}^2$ (or, more generally, of an algebraic variety) influence the set $C \cap \mathbb{Q}^2$, assuming that C can be defined by equations with coefficients in $\mathbb{Q}$.

4. **Hypersurfaces.** A hypersurface is by definition $V(f) \subset K^n$ where $f \in K[x_1, \dots, x_n]$ is an irreducible polynomial. These are heavily studied for many reasons: they are easy to define, yet they already give very rich families of examples (in fact, as we shall see, every variety is birational to some hypersurface).

1.1.2 Localization and open subsets

Let us now recall the operation of localization and how to describe the Zariski open subsets of a variety $X \subset K^n$. Let R be any commutative ring and let $S \subset R$

be a multiplicative subset. We defined the localization $S^{-1}R$ last year: this is a ring together with a natural map

$$\phi : R \to S^{-1}R$$

which is universal among ring maps $f : R \to R'$ such that $f(S) \subset (R')^\times$, i.e. every $f(s)$ is invertible for $s \in S$. We have shown that the induced map

$$\mathrm{Spec}(S^{-1}R) \longrightarrow \mathrm{Spec}(R)$$

is an open immersion with image

$$\{\mathfrak{p} \in \mathrm{Spec}(R) : S \cap \mathfrak{p} = \varnothing\}.$$

The two most important multiplicative sets in algebraic geometry are:

- $S = \{1, f, f^2, \dots\}$ for some $f \in R$, in which case we write $S^{-1}R = R_{(f)}$;
- $S = R \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, in which case we write $S^{-1}R = R_{\mathfrak{p}}$.

Recall that if R is a domain, then $S^{-1}R \subset \mathrm{Frac}(R)$ via the universal property of localization. Concerning $R_{(f)}$, we have a natural identification

$$R_{(f)} \cong \{\mathrm{Frac} h f^n : h \in R,\ n \geq 0\} \subset \mathrm{Frac}(R).$$

Now let $A = K[x_1, \dots, x_n]/\mathfrak{p}$ and let $X = V(\mathfrak{p}) \subset K^n$ be the associated algebraic variety. For any $f \in A$ consider the open subset $D_f = X \setminus V(f)$. These are called *principal open subsets.*

Lemma 1.8. *The following hold:*

1. *The open sets D_f generate the Zariski topology of X;*
2. *Every D_f has a natural structure of an affine algebraic variety.*

Proof. Point (1) is an exercise. For (2), write $\mathfrak{p} = (f_1, \dots, f_k) \subset K[x_1, \dots, x_n]$, so that

$$R_{(f)} \cong K[x_1, \dots, x_n, y]/(f_1, \dots, f_k, yf - 1).$$

Thus $R_{(f)}$ is itself a finitely generated K-algebra, and hence defines an affine algebraic variety in K^{n+1}. Its points correspond uniquely to maximal ideals of $R_{(f)}$, and hence to points of D_f. □

In general, if $X \subset K^n$ is an algebraic variety and $U \subset X$ is an open subset, we say that $U \subset K^n$ is a quasi affine variety.

Excercise 1.9. Show that $\mathrm{GL}_n(K) \subset M_n(K)$ has a natural structure of an affine algebraic variety, where we identify $M_n(K) \cong K^{n^2}$.

1.2 Dimension theory

Recall the definition of the Krull dimension of a ring R: it is the maximal length of a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

(if such a maximum exists). Similarly, for a topological space X, we define $\dim(X)$ as the maximal length of a chain of non-empty irreducible closed subsets

$$Z_n \subsetneq \cdots \subsetneq Z_1 \subsetneq Z_0.$$

So if $A = K[x_1, \cdots, x_n]/I$ is a finitely generated K-algebra $\dim(A) = \dim(V(I))$. We also proved that if A is an integral finitely generated K-algebra then

$$\dim(A) = \text{tr.deg.}_K(\text{Frac}(A)).$$

One would like to have a well-defined notion of codimension. Clearly we could just define $\text{codim}(Z) = \dim(X) - \dim(Z)$ where $Z \subset X$ is a closed subset, but this is an ad-hoc definition which is not well-behaved in practice. The only thing we can do algebraically is the following:

Definition 1.10. Let $\mathfrak{p} \subset R$ be a prime. Then its height $\text{ht}(\mathfrak{p})$ is the maximal length of a chain of prime ideals $\mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}$ (if it exists).

The problem now is to show that

$$\text{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R).$$

But this is not true for general rings, and the right notion to consider is the following:

Definition 1.11. A ring R is *catenary* if for any two primes $\mathfrak{p}_1 \subset \mathfrak{p}_2$, all maximal chains of prime ideals starting at $\mathfrak{p}_1$ and ending at $\mathfrak{p}_2$ have the same finite length.

It is easy to see that any localization or quotient of a catenary ring is still catenary (prove it). The first counterexample of a Noetherian ring which is not catenary was constructed by Nagata (he also constructed the first example of a Noetherian ring with infinite Krull dimension). On the other hand, most if not all of the rings appearing in algebraic geometry are catenary:

Theorem 1.12. *If A is a finitely generated integral K-algebra then it is catenary.*

Proof. The proof can be found in Matsumura's book. We advise the reader to take a look at it. □

Note that without the integrality condition the result clearly fails. For example, consider

$$A = K[x, y, z]/(xy, xz).$$

Then A is not integral (picture it in K^3). One can check that both chains

$$(y, z) \subset (x, y, z), \qquad (x) \subset (x, y) \subset (x, y, z)$$

are maximal chains of prime ideals of different lengths. What is happening geometrically?

Proposition 1.13. *Let R be integral, catenary, and of finite dimension. Then for any prime $\mathfrak{p} \subset R$ we have*

$$\dim(R) = \mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}).$$

Proof. The proof follows by concatenating chains of prime ideals starting with (0) and ending with $\mathfrak{p}$ with chains of prime ideals in $R/\mathfrak{p}$, and using catenarity to ensure equality of lengths. □

We shall return to the notion of dimension again later in the course. For the time being, consider the following:

Example 1.14. Let $(R, \mathfrak{m})$ be a discrete valuation ring with uniformizer $\pi \in \mathfrak{m}$ fraction field K and residue field k. Consider the polynomial ring $R[x]$. Then, there are two kind of maximal ideals $m \subset R[x]$:

1. Either $m \cap R = \mathfrak{m}$ or
2. $m \cap R = (0)$.

For example, the ideal $(\pi, x - r)$ for $r \in R$ belongs to case (1) whereas $(\pi x - 1) \subset R[x]$ to case (2). In particular $R[x]$ is not catenary.

1.3 Morphisms of varieties

Every time we define new objects in mathematics we also have to define morphisms between them. We have two ways to do so in the case of affine algebraic varieties: let $X \subset K^n$ and $Y \subset K^m$ be two varieties:

1. A map $f : X \to Y$ is a morphism if there are $P_1, \cdots, P_m \in K[x_1, \cdots, x_n]$ such that $f(x) = (P_1(x), \cdots, P_m(x))$ for every $x \in X$.
2. A map $f : X \to Y$ is a morphism if for every $P \in X$ there are polynomial functions $H, G : K^n \to K^m$ such that $G(u) \neq 0$ and $f(u) = H(u)/G(u)$ for every $u \in U \cap X$.

The two definitions turn out to be equivalent, but a priori the first definition is only contained in the second, and it is more rigid as it forces all morphisms to be restrictions of 'global' polynomial maps between K^n and K^m. For example, the first definition does not work for quasi-affine subsets, but the second does. Thinking more about the difference between the two definitions naturally leads to the notion of sheaves. We adopt the first definition for the time being. Finally, we say that X and Y are isomorphic if there are morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.

- Show that any $f \in \mathrm{Hom}(X, Y)$ is continuous for the Zariski topology.

- Since K is a ring $\mathrm{Mor}(X, K)$ is also a ring. Show that $\mathrm{Mor}(X, K) \cong A$ where $X = V(I) \subset K^n$ and $A = K[x_1, \cdots, x_n]/\sqrt{I}$.

- In particular, take $f \in A$. Then $\mathrm{Mor}(D_f, K) \cong R_{(f)}$, where we give D_f the structure of an algebraic variety as before.

- (Linear projections) Let W be a finite dimensional K-vector space. Let $\phi : K^n \to W$ be a linear surjective morphism and let $X \subset K^n$ be an algebraic variety. Then the restriction $\phi_{|X}$ is called a linear projection from X to W.

- (Projections from a point) Choose a point $P \in K^n$ and a linear hyperplane $H \subset K^n$ (that is, the translate of a linear subspace $V \subset K^n$ of dimension $n - 1$ or, equivalently, the zero set of a degree one polynomial $F \in K[x_1, \cdots, x_n]$) such that $P \notin H$. For any point $x \in K^n \setminus P$ let ℓ_x be the line joining P and x. Then only two things can happen: either $\ell_x \cap H$ consists of one point or $\ell_x \cap H = \emptyset$. This second case happens only if $x \in H'$, the unique hyperplane parallel to H with $P \in H'$. So on $D_{H'} = K^n \setminus H'$ we obtain a well-defined set-theoretic map $D_{H'} \to H$, called the projection from P to H. Show that this is a morphism of algebraic varieties. We can restrict this to any algebraic variety and obtain a morphism defined on a principal open subset.

- (Affine conics, continued). Consider an irreducible conic $C = V(Q) \subset K^2$ and assume that $(0, 0) \in C$, which is always possible up to translation. We want to show that either C is isomorphic to K or to $K \setminus 0 = D_x$ as algebraic varieties. In fact, we can do this in two ways. To prove the result algebraically, write $Q = Q_2 + Q_1$ where Q_2 is homogeneous of degree 2 and Q_1 is homogeneous of degree 1 (there is no constant term since $(0, 0) \in Q$). So $Q_2(x, y) = ax^2 + bxy + cx^2$ which is the quadratic form associated to the bilinear form $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ (recall that char$(K) \neq 2$). Then Q_2 as a matrix has rank either one or two. With a linear change of variable, show that in the first case C is isomorphic to the parabola $V(y - x^2) \cong K$ and that in the second one to the hyperbole $V(xy - 1) \cong K \setminus 0$.

One can also prove the result geometrically, but the light on this will be shed once we introduce projective spaces, and see the conic as a curve inside the projective plane. One begins by answering the following: how many lines $L \subset K^2$ passing through $(0, 0)$ are such that $L \cap C = \{(0, 0)\}$? Clearly the tangent line $T \subset K^2$ is one of them. Show that there are either one or two more, and fix one of them and call it L. Finally, for any point $P' \in C$ let L' be the unique line parallel to L and passing through P'. Show that $L' \cap T$ consists of a unique point $\phi(P')$. Show that this defines a morphism $\phi : C \to T$. Show that in the first case (where there is only one line L) the map is an isomorphism, and in the second, the map is an isomorphism onto the complement of a point.

- (Frobenius) Assume that $\mathrm{char}(K) = p > 0$ and let $X = V(\mathfrak{p}) \subset K^n$ be a variety. Assume for simplicity that $\mathfrak{p} = (f_1, \cdots, f_d)$ where $f_i \in \mathbb{F}_p[x_1, \cdots, x_n] \subset K[x_1, \cdots, x_n]$ for every i. Now consider the map $F : K^n \to K^n$ sending $(a_1, \cdots, a_n) \mapsto (a_1^p, \cdots, a_n^p)$, which corresponds to the K-algebra map $K[x_1, \cdots, x_n] \to K[x_1, \cdots, x_n]$ sending $x_i \mapsto x_i^p$. Show that F induces a bijection $F : X \to X$, which is called the geometric Frobenius of X. Note that we would expect the map $F : X \to X$ to have degree $p^{\dim(X)}$ (for instance, try to show that the induced map of function fields $\mathrm{Frac}(A(X)) \to \mathrm{Frac}(A(X))$ yields a finite extension of degree $p^{\dim(X)}$) but topologically the map is a bijection, hence it should be an isomorphism. This 'problem' is also solved by schemes, for which indeed the map turns out to have the right degree.

Now, let $X \subset K^n$ be given by $V(\mathfrak{p})$ and $Y \subset K^m$ be given by $V(\mathfrak{q})$.

Proposition 1.15. *There is a natural identification*

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}_{\text{K-}alg}(A(Y), A(X))$$

where, from now on, we write $A(X) = \mathrm{Hom}(X, K)$.

Proof. Take a morphism $f : X \to Y$. Then the induced map $A(Y) \to A(X)$ is simply given by composition

$$A(Y) \ni (g : Y \to K) \mapsto (g \circ f : X \to K) \in A(X),$$

which defines a ring morphism.

For the other direction, write $A(X) = K[x_1, \cdots, x_n]/\mathfrak{p}$ and $A(Y) = K[x_1, \cdots, x_m]/\mathfrak{q}$. Consider a map of rings $\phi : A(Y) \to A(X)$; this induces a map of rings $K[x_1, \cdots, x_m] \to A(X)$ which sends x_i to some $f_1' \in A(X)$. Now $A(X)$ is quotient of $K[x_1, \cdots, x_n]$. Find $f_i \in K[x_1, \cdots, x_n]$ which reduce to $f_i' \in A(X)$; by freeness of $K[x_1, \cdots, x_n]$ we can thus define define a morphism $K[x_1, \cdots, x_m] \to K[x_1, \cdots, x_n]$ hence a morphism $f : K^n \to K^m$ sending $(k_1, \cdots, k_n) \mapsto (f_1(k_1, \cdots, k_n), \cdots, f_m(k_1, \cdots, k_n))$. The facet that $f(X) \subset Y$ then follows from the commutativity of

$$\begin{tikzcd}
K[x_1, \cdots, x_m] \arrow[r] \arrow[d, two heads] & K[x_1, \cdots, x_n] \arrow[d, two heads] \\
A(Y) \arrow[r] & A(X)
\end{tikzcd}$$

These constructions are one the inverse of the other. $\square$

2 Lecture II / III

2.1 Presheaves

The philosophy behind sheaves comes precisely from the two different possible definitions of morphisms between algebraic varieties. Although the concept of sheaf is not always stated explicitly, it is already present in many constructions in analysis. For example, let $U \subset \mathbb{C}^n$ be an open subset (with the usual topology) and let $f : U \to \mathbb{C}$ be a continuous function. Then f is said to be holomorphic on U if, for every point $P \in U$, there exists an open neighborhood $V \subset U$ of P over which f can be expressed as a converging power series. In other words, a holomorphic function is defined by its *local behaviour*, and typically we cannot express f globally on all of U as a single converging power series. This illustrates the principle that global objects are determined by compatible local data, which is exactly what the concept of a sheaf formalizes.

Definition 2.1. Let X be a topological space. A *presheaf* of abelian groups $\mathscr{F}$ on X consists of the following data:

1. For every open subset $U \subset X$, an abelian group $\mathscr{F}(U)$;
2. For every inclusion of open sets $V \subset U$, a *restriction map*

$$\rho_{U,V} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$$

   satisfying the following:

   (a) $\rho_{U,U} = \mathrm{Id}_{\mathscr{F}(U)}$;

   (b) For any chain $U_1 \subset U_2 \subset U_3$, we have $\rho_{U_2,U_1} \circ \rho_{U_3,U_2} = \rho_{U_3,U_1}$.

An element $s \in \mathscr{F}(U)$ is called a *section* of $\mathscr{F}$ over U, and we write

$$s|_V := \rho_{U,V}(s)$$

for its restriction to $V \subset U$. The group $\mathscr{F}(X)$ is often denoted $\Gamma(X, \mathscr{F})$ and its elements are called *global sections*.

Remark 2.2. Similarly, one can define presheaves of sets, rings, or other objects. In fact, for any category $\mathscr{C}$, a presheaf with values in $\mathscr{C}$ can be defined in the same way. Equivalently, let $\mathrm{Open}(X)$ be the category whose objects are open subsets of X, with

$$\mathrm{Hom}(U, V) = \begin{cases} \{*\}, & \text{if } U \subset V, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Then a presheaf with values in $\mathscr{C}$ is exactly a contravariant functor

$$\mathscr{F} : \mathrm{Open}(X) \longrightarrow \mathscr{C},$$

where $\mathscr{F}(U)$ is the value on U and the restrictions are given by the functorial action.

2.1.1 Examples

We now make some important examples of presheaves, divided into two classes.

1. Structure sheaves of geometric objects.

   All the following are presheaves of rings, with restriction maps given by restricting functions to smaller open sets.

   (a) *Continuous functions:* Let X be a topological space. Define

   $$\mathscr{C}_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is continuous}\}.$$

   (b) *Differentiable functions:* Let X be a $\mathscr{C}^\infty$-manifold. Define

   $$\mathscr{C}^\infty_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is smooth}\}.$$

   (c) *Holomorphic functions:* Let X be a complex manifold. Define

   $$\mathscr{O}^{\mathrm{hol}}_X(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}.$$

   (d) *Regular functions:* Let $X \subset K^n$ be an algebraic variety with the Zariski topology. Define

   $$\mathscr{O}_X(U) = \left\{ f : U \to K \mid \begin{array}{c} \text{for every } P{\in}U \text{ there is an open } P{\in}V{\subset}U \text{ and} \\ \text{polynomials } h,g{\in}K[x_1,\cdots,x_n] \text{ such that} \\ g(v){\neq}0 \text{ and } f(v){=}h(v)/g(v) \text{ for every } v{\in}V \end{array} \right\}.$$

2. Constant presheaves.

   (a) Let G be an abelian group. The constant presheaf with values in G is

   $$\underset{\sim}{G}(U) = \{f : U \to G \mid f \text{ is constant}\}.$$

   (b) Endow G with the discrete topology. The *locally constant presheaf* $\underline{G}$ is defined by

   $$\underline{G}(U) = \{f : U \to G \mid f \text{ is continuous}\}.$$

(c) Let $\pi : Y \to X$ be a covering map of topological spaces. Define the presheaf of sections

$$\mathcal{S}_{\mathfrak{p}}i(U) = \{f : U \to Y \mid f \text{ continuous and } \pi \circ f = \mathrm{Id}_U\}.$$

We now discuss some differences and similarities between the examples of presheaves introduced before. The first and most important observation is that all the presheaves in point (1) satisfy some extra fundamental properties.

Let $\mathcal{O}$ be any of the presheaves from point (1), let $U \subset X$ be an open subset, and let $\{U_i\}_{i \in I}$ be any open cover of U. Then the following hold:

I. **(Gluing of functions)** Given sections $f_i \in \mathcal{O}(U_i)$ such that

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \quad \text{for all } i, j \in I,$$

there exists a section $f \in \mathcal{O}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

II. **(Local nature of functions)** If $f, g \in \mathcal{O}(U)$ are such that

$$f|_{U_i} = g|_{U_i} \quad \text{for all } i \in I,$$

then $f = g$.

These two properties are exactly what distinguish *sheaves* from general presheaves. Let us see how these properties can fail:

Failure of gluing (I). Consider a topological space $X = X_1 \sqcup X_2$ with $X_1, X_2 \neq \emptyset$, and let $\tilde{G}$ be the constant presheaf defined earlier. Then

$$\tilde{G}(X_1) = G, \quad \tilde{G}(X_2) = G, \quad \tilde{G}(\emptyset) = 0.$$

Any two sections $g_1 \in \tilde{G}(X_1)$ and $g_2 \in \tilde{G}(X_2)$ satisfy the local compatibility condition, but if $g_1 \neq g_2$, they cannot be glued to a global section in $\Gamma(X, \tilde{G})$. In contrast, the locally constant presheaf $\underline{G}$ satisfies the gluing property and we have $\underline{G}(X) = G^{\pi_0(X)}$ (at lease when all connected components of X are open).

Failure of locality (II). To construct a presheaf that fails property (II) is slightly more subtle. Consider the presheaf of bounded continuous functions

$$\mathcal{C}^b_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is bounded}\} \subset \mathcal{C}_X(U).$$

Then the quotient presheaf

$$\mathcal{F}(U) = \mathcal{C}_X(U)/\mathcal{C}^b_X(U)$$

does not satisfy (II). For example, if $X = \mathbb{R}$, consider the section represented by $x \in \mathcal{C}_X(\mathbb{R})$. Then $0 \neq x \in \mathcal{F}(\mathbb{R})$, but for any open cover $\{U_i\}_{i \in I}$ of $\mathbb{R}$ such that each $\overline{U_i}$ is compact we have $x|_{U_i} = 0$ in $\mathcal{F}(U_i)$.

2.2 Sheaves

Definition 2.3. A presheaf $\mathscr{F}$ is called a *sheaf* if for every open subset $U \subset X$ and every open cover $\{U_i\}_{i\in I}$ of U, the following hold:

I. **(Gluing of sections)** Given sections $s_i \in \mathscr{F}(U_i)$ such that

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \quad \text{for all } i, j \in I,$$

there exists a section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

II. **(Locality)** If $s, s' \in \mathscr{F}(U)$ satisfy

$$s|_{U_i} = s'|_{U_i} \quad \text{for all } i \in I,$$

then $s = s'$.

Remark 2.4. Property (II) guarantees that the gluing in (I) is unique.

Now, there is a canonical way to obtain a sheaf from a presheaf but first we need talk about stalks (note: the terminology sheaves and stalks is not random). Stalks are used to study sections around a specific point $P \in X$:

Definition 2.5. Let $\mathscr{F}$ be a presheaf and let $P \in X$. The *stalk* of $\mathscr{F}$ at P is

$$\mathscr{F}_P = \varinjlim_{P\in U} \mathscr{F}(U),$$

where the colimit runs over all open neighborhoods U of P. An element of $\mathscr{F}_P$ is called a *germ* of a section at P. Similarly, for a subset $Z \subset X$, the stalk at Z is defined by

$$\mathscr{F}_Z = \varinjlim_{Z\subset U} \mathscr{F}(U).$$

Remark 2.6. In practice, a germ in $\mathscr{F}_P$ is represented by a pair (U, s) with $P \in U$ and $s \in \mathscr{F}(U)$. Two such pairs (U, s) and (U', s') define the same germ if there exists an open $V \subset U \cap U'$ containing P such that $s|_V = s'|_V$. If $\mathscr{F}$ takes values in a category $\mathscr{C}$, we assume that $\mathscr{C}$ admits direct limits so that this definition makes sense.

Remark 2.7. For the sheaves of functions introduced earlier, the restriction maps $\rho_{U,V}$ have special properties: they are surjective for $\mathscr{C}_X$ and $\mathscr{C}_X^\infty$ (due to the existence of partitions of unity), and injective for $\mathscr{O}_X^{\text{hol}}$ and $\mathscr{O}_X$ (because holomorphic and regular functions are rigid: if they agree on an open subset, they agree everywhere).

Let now $X = V(\mathfrak{p}) \subset K^n$ be a variety, and consider the sheaf $\mathcal{O}_X$ more closely. Let $K(X) = \mathrm{Frac}(A(X))$.

Lemma 2.8. *For any open subset $U \subset X$, there is a natural inclusion*

$$\mathcal{O}_X(U) \subset K(X).$$

Proof. Let $f \in \mathcal{O}_X(U)$. By definition, there exists a finite open cover U_i of U and polynomials $P_i, Q_i \in K[x_1, \dots, x_n]$ such that $Q_i(u) \neq 0$ for all $u \in U_i$ and $f(u) = P_i(u)/Q_i(u)$ on U_i. Now $Q_i \notin \mathfrak{p}$ for otherwise $Q_i(x) = 0$ for every $x \in X$. Hence Q_i is invertible in the localization $K[x_1, \cdots, x_n]_{\mathfrak{p}}$ and so it defines an element $[Q_i] \in K(X) = K[x_1, \cdots, x_n]_{\mathfrak{p}}/\mathfrak{p}$. We claim that $[P_i][Q_i]^{-1} \in K(X)$ is well-defined and independent of i. First, if $P_i(u)/Q_i(u) = P_i'(u)/Q_i'(u)$ for every $u \in U_i$ then $P_iQ_i' - P_i'Q_i$ defines an element of $A(X)$ whose vanishing locus contains U_i, which is a non-trivial Zariski open. If $P_iQ_i' - P_i'Q_i$ was non-zero in $A(X)$ then also its non-vanishing locus would be a non-empty Zariski open. Since any two Zariski opens intersect because X is irreducible, we see that $P_iQ_i' - P_i'Q_i \in \mathfrak{p}$ and thus $[P_i][Q_i]^{-1} \in K(X)$ does not depend on the representatives P_i, Q_i over U_i. Similarly (i.e., using the fact that $U_i \cap U_j \neq \emptyset$) one shows that $[P_i][Q_i]^{-1} = [P_j][Q_j]^{-1}$ as well.

This defines a map $\mathcal{O}_X(U) \to K(X)$. Suppose that this were not injective. Then $[P_i] \in A(X)$ must vanish on a non-trivial Zariski open - hence it vanishes everywhere - and therefore $f = 0$. □

Remark 2.9. This also shows that the restriction maps $\rho_{U,V}$ for $\mathcal{O}_X$ are injective whenever $V \neq \emptyset$.

Theorem 2.10. *Let $X = V(\mathfrak{p}) \subset K^n$ be as before.*

1. *For every $x \in X$ the stalk $\mathcal{O}_{X,x}$ is isomorphic to the localization $A(X)_{\mathfrak{m}_x}$.*
2. *We can identify*
$$\mathcal{O}_X(U) = \bigcap_{x \in U} A(X)_{\mathfrak{m}_x} \subset K(X)$$
3. *In particular, $\Gamma(X, \mathcal{O}_X) = A(X)$ (as promised).*

Proof.

- Using the previous lemma we have
$$\mathcal{O}_{X,x} = \{f/g : f, g \in A(X) \text{ and } g(x) \neq 0\} \subset K(X).$$
- The inclusion $\mathcal{O}_X(U) \subset \bigcap_{x \in U} A(X)_{\mathfrak{m}_x}$ is clear. Let now h be in the intersection, then for every $x \in U$ we can write $f = h_x/g_x$ for some

$f_x, g_x \in A(X)$ such that $g_x(x) \neq 0$. So g_x does not vanish in a neighbrhood $x \in U_x \subset X$ of x. But then there are finitely many $x_1, \cdots, x_n$ such that $U_i = U_{x_i}$ cover X and such that $g_i(u) \neq 0$ for every $u \in U_i$. Thus, the function $U \to K$ sending $u \mapsto f_i(u)/g_i(u)$ if $u \in U_i$ is well-defined and defines a section of $\mathcal{O}_X(U)$.

- Let $h \in \bigcap_{x \in X} A(X)_{\mathfrak{m}_x}$ and consider the ideal $I = \{g \in A(X) : gh \in A(X)\} \subset A(X)$. If this is not trivial, then it is contained in a maximal ideal $\mathfrak{m} \subset A(X)$ corresponding to some $x \in X$. But $h \in A(X)_{\mathfrak{m}}$ so $h = f/g$ with $f \in A(X)$ and $g(x) \neq 0$ and hence also $g \in I$, contradicting that $I \subset \mathfrak{m}$.

□

The fact that a sheaf is determined by its local behavious is captured in a number of propositions, for example:

Proposition 2.11. *If $\mathcal{F}$ is a sheaf then for every open $U \subset X$ the natural map*

$$\mathcal{F}(U) \to \prod_{P \in U} \mathcal{F}_P$$

is injective.

Proof. If a $s \in \mathcal{F}(U)$ is in the kernel of the above, then by the definition of stalks, we can find for every point P an open $P \in U_P$ such that $s_{|U_P} = 0$. Since $\{U_P\}_P$ covers X the result follows by condition (II). □

Note that in fact one can also characterize the image:

Proposition 2.12. *If $\mathcal{F}$ is a sheaf then the image of*

$$\mathcal{F}(U) \to \prod_{P \in U} \mathcal{F}_P$$

corresponds to

$$\left\{ (s_P) \in \prod_{P \in U} \mathcal{F}_P : \begin{array}{c} \textit{for every } P \textit{ there is } V \subset U \textit{ open with} \\ P \in V \textit{ and a section } S \in \mathcal{F}(V) \textit{ such} \\ \textit{that } s_Q = S_Q \textit{ in } \mathcal{F}_Q \textit{ for every } Q \in V \end{array} \right\}.$$

Proof. Use the previous proposition and the glueing condition. □

Sheafification So now it is easy to associate a sheaf to any presheaf

Definition 2.13. If $\mathscr{F}$ is a presheaf its sheafification $\mathscr{F}^+$ is defined by

$$\mathscr{F}^+(U) := \left\{ (s_P) \in \prod_{P\in U} \mathscr{F}_P : \begin{array}{c} \text{for every } P \text{ there is } U{\subset}X \text{ open with} \\ P{\in}U \text{ and a section } S{\in}\mathscr{F}(U) \text{ such} \\ \text{that } s_Q{=}S_Q \text{ in } \mathscr{F}_Q \text{ for every } Q{\in}U \end{array} \right\}.$$

This is a sheaf and there is a natural map of presheaves $\mathscr{F} \to \mathscr{F}^+$ which is an *isomorphism* on stalks. It satisfies a universal property: for any sheaf $\mathscr{G}$ any morphism $\mathscr{F} \to \mathscr{G}$ factorises uniquely as $\mathscr{F} \to \mathscr{F}^+ \to \mathscr{G}$.

Remark 2.14. To get another glimpse into the local nature of sheaves, consider the following example: let $\mathscr{P}$ be a property on the open subsets of X such that if $U \subset V$ and $\mathscr{P}(V)$ is true then also $\mathscr{P}(U)$ is true. Define the presheaf of sets $\mathscr{F}(U) = \{*\}$ if $\mathscr{P}(U)$ is true and $\mathscr{F}(U) = \emptyset$ if $\mathscr{P}(U)$ is false and let $\mathscr{F}^+$ be its sheafification. Then $\Gamma(X, \mathscr{F}) = \{*\}$ if and only if $\mathscr{P}(X)$ is true whereas $\Gamma(X, \mathscr{F}^+) = \{*\}$ if and only if $\mathscr{P}$ is *locally true* over X, that is, X can be covered by open subsets U_i such that $\mathscr{P}(U_i)$ is true for every i.

2.3 Exact properties of sheaves and presheaves

Definition 2.15. Let $\mathscr{F}, \mathscr{G}$ be presheaves on X of sets, abelian groups, etc. A morphism $\phi : \mathscr{F} \to \mathscr{G}$ is by definition a collection of maps of sets, abelian groups, etc $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ for every open $U \subset X$ satisfying the obvious compatibility with the restriction maps.

1. ϕ is injective if ϕ_U is injective for every U;
2. ϕ is surjective if ϕ_U is surjective for every U.

Assume now that both $\mathscr{F}$ and $\mathscr{G}$ are presheaves of abelian groups:

1. $\ker(\phi)$ is the presheaf $U \mapsto \ker(\phi_U)$;
2. $\operatorname{coker}(\phi)$ is the presheaf $U \mapsto \operatorname{coker}(\phi_U)$
3. $\operatorname{Im}(\phi)$ is the presheaf $U \mapsto \operatorname{Im}(\phi_U)$.

Remark 2.16. One can form a category of presheaves of abelian groups on X which turns out to be an abelian category. The definitions above then corresponds to the abstract definitions of the same objects in any abelian category.

Remark 2.17. Recall that direct limits preserve exactness. In particular, if $\phi : \mathscr{F} \to \mathscr{G}$ is injective/surjective then the induced map $\phi_P : \mathscr{F}_P \to \mathscr{G}_P$ is injective/surjective for every point P.

For sheaves, the situation is different. One can again form the category of sheaves of abelian groups on X, which again turns out to be an abelian category. But it might be that $\mathscr{F} \to \mathscr{G}$ is an epimorphism of sheaves without being an epimorphism of presheaves:

Proposition 2.18 (Exactness properties of sheaves). *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. Then:*

1. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves also the presheaf $\ker(\phi)$ is a sheaf;*
2. *If ϕ_U is injective for every open U then also $\phi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ is injective;*
3. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves then the sheafification of $\operatorname{Im}(\phi)$ is naturally a subsheaf of $\mathscr{G}$;*
4. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves then ϕ is an epimorphism of sheaves if and only if ϕ_P is surjective for every P.*

Proof.

1. Clearly $\ker(\phi)$ satisfies (II), so let us show that we can glue sections. Let $\bigcup_i U_i = U$ be an open cover of some open $U \subset X$ and let $s_i \in \ker(\phi_{U_i})$ be local sections. Assume that they satisfy the glueing condition. Then, since $\mathscr{F}$ is a sheaf we can at least find some $s \in \mathscr{F}(U)$ such that $s_{|U_i} = s_i$ where equality is taken inside $\mathscr{F}(U_i)$. So consider $\phi(s) \in \mathscr{G}(U)$. But $\phi(s)_{|U_i} = \phi(s_{|U_i}) = \phi(s_i) = 0$. Since $\mathscr{G}$ is a sheaf we then have $\phi(s) = 0$, i.e., $s \in \ker(\phi_U)$.
2. By construction of the sheafification, it is sufficient to show that the induced map of stalks $\phi_P : \mathscr{F}_P \to \mathscr{G}_P$ is injective for every P, which follows by the previous remark.
3. Since the map of presheaves $\operatorname{Im}(\phi) \to \mathscr{G}$ is injective by definition also $\operatorname{Im}(\phi)^+ \to \mathscr{G}^+ = \mathscr{G}$ is injective by the previous point.
4. In fact, ϕ is an epimorphism if and only if $\operatorname{Im}(\phi)^+ = \mathscr{G}$. By what we have shown, this is equivalent to $\operatorname{Im}(\phi)^+_P = \mathscr{G}_P$ for every point P. But $\operatorname{Im}(\phi)^+_P = \operatorname{Im}(\phi_P) = \mathscr{G}_P$ by assumption.

□

Now, in any abelian category, if a morphism is both an epimorphism (surjective) and a monomorphism (injective) it must be invertible. Let us verify this:

Proposition 2.19. *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphim of sheaves (not necessarily of abelian groups) such that ϕ_P is an isomorphism for every P. Then ϕ is an isomorphism.*

Proof. We know that $\phi_{|U}$ is injective for every $U \subset X$ open. We only need to show that it is also surjective. So pick $s \in \mathscr{G}(U)$ and let $P \in U$. Then we can find by assumption some $t_P \in \mathscr{F}_P$ such that $\phi_P(t_P) = s_P$. Let t_P be represented by $t'_P \in \mathscr{F}(V'_P)$ where $P \in V'_P \subset U$ is some open subset. Then the various V'_P cover X. Moreover, the various t'_P agree on the overlappings $V'_P \cap V'_Q$ due to the injectivity of ϕ, implying that they satisfy the glueing condition - hence the result. □

Returning to our discussion, the fact that surjectivity for sheaves is a local condition naturally leads to sheaves cohomology: consider a short exact sequence of abelian sheaves

$$0 \to \mathscr{F}_1 \xrightarrow{f} \mathscr{F}_2 \xrightarrow{g} \mathscr{F}_3 \to 0$$

meaning that $\mathscr{F}_1 = \ker(g)$ and $\mathscr{F}_3 = \operatorname{coker}(f)$. Then

$$0 \to \Gamma(X, \mathscr{F}_1) \xrightarrow{f_X} \Gamma(X, \mathscr{F}_2) \xrightarrow{g_X} \Gamma(X, \mathscr{F}_3)$$

is exact, but g_X need not be surjective. So $\Gamma(X, -)$ is a left-exact functor and one we study its derived functors, whose role in this case is to measure the obstructions that prevent local data from glueing into global sections. From the behaviour of such cohomology groups we then get information about the geometry of X. This will be done in the second part of the course. Let us see now a nice example which mixes complex analysis and topology:

Example 2.20. Let $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}^\times \subset \mathbb{C}$ be the circle group with its usual topology. Let $\mathscr{C}_{\mathbb{S},\mathbb{C}}$ be the sheaf of continuous functions from $\mathbb{S}$ to $\mathbb{C}$ and let $\mathscr{C}^\times_{\mathbb{S},\mathbb{C}}$ be the sheaf of continuous invertible functions from $\mathbb{S}$ to $\mathbb{C}^\times$. Then we have a short exact sequence of sheaves (verify it)

$$0 \to (2\pi i)\underline{\mathbb{Z}} \to \mathscr{C}_{\mathbb{S},\mathbb{C}} \xrightarrow{\exp} \mathscr{C}^\times_{\mathbb{S},\mathbb{C}} \to 0$$

where $\underline{\mathbb{Z}}$ is the locally constant sheaf and exp sends a local function $g : U \to \mathbb{C}$ to $\exp(g)$, where $U \subset \mathbb{S}$. Let now $f : \mathbb{S} \to \mathbb{C}^\times$ be a global section $f \in \Gamma(\mathbb{S}, \mathscr{C}^\times_{X,\mathbb{C}})$. By fixing $1 \in \mathbb{S}$ we can consider f as a continuous loop in $\mathbb{C}^\times$ with base point $f(1)$, and so f yields an element $[f] \in \pi_1(\mathbb{C}^\times, f(1)) \cong \mathbb{Z}$. Show that $f = \exp(g)$ for some global section $g : \mathbb{S} \to \mathbb{C}$ if and only if $[f] = 0$.

2.4 Functoriality of sheaves and ringed spaces

Let X, Y be topological spaces, $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y and $f : X \to Y$ a continuous map.

1. Push-forward (or direct image) One defines the sheaf $f_*\mathscr{F}$ on Y by the rule $U \mapsto \mathscr{F}(f^{-1}(U))$. Show that this is a sheaf.
2. Inverse image One defines the sheaf $f^{-1}\mathscr{G}$ on X as the sheafification of $U \mapsto \mathscr{G}_{f(U)}$ (the stalk of $\mathscr{G}$ at $f(U)$).

Note that although f_* is easier to define, it is easier to compute the stalks of f^{-1}:

Proposition 2.21. *For $x \in X$ we have natural isomorphisms:*

1. $(f_*\mathscr{F})_{f(x)} \cong \mathscr{F}_{f^{-1}(f(x))}$.
2. $(f^{-1}\mathscr{G})_x = \mathscr{G}_{f(x)}$.

Proof. This follows directly from the definition. □

In particular, f^{-1} is exact and f_* is left-exact. We shall define f^* later in the context of schemes. Finally, suppose we are in the situation of the previous definition, and that we want to define a morphism of sheaves $\mathscr{G} \to \mathscr{F}$ 'along f'. We can then either consider $\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$ or $\mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$: the two are the same.

Proposition 2.22 (adjunction inverse/direct image). *There is a natural isomorphism* $\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$.

Proof. We construct maps in both directions. Take $\phi \in \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$, so for every open $V \subset Y$ we have a compatible set of morphisms $\phi_V : \mathscr{G}(V) \to \mathscr{F}(f^{-1}(V))$. Pick any open $U \subset X$ and suppose that $U \subset f^{-1}(V)$ (which is equivalent to $f(U) \subset V$). Then we get a map

$$\mathscr{G}(V) \xrightarrow{\phi_V} \mathscr{F}(f^{-1}(V)) \xrightarrow{\rho_{f^{-1}(V),U}} \mathscr{F}(U)$$

and this is easily seen to define a morphism

$$(f^{-1}\mathscr{G})(U) = \varinjlim_{f(U)\subset V} \mathscr{G}(V) \to \mathscr{F}(U)$$

and thus an element of $\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$. To obtain a map in the other direction, pick $\psi \in \mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$ and let $V \subset Y$ be any open subset. Put $U = f^{-1}(V)$, which is open in X. So by definition of direct limit we have a map $\mathscr{G}(V) \to f^{-1}(\mathscr{G})(U)$ and hence we obtain a morphism

$$\mathscr{G}(V) \to f^{-1}(\mathscr{G})(U) \xrightarrow{\psi_U} \mathscr{F}(U) = \mathscr{F}(f^{-1}(V)) = (f_*\mathscr{F})(V)$$

hence an element of $\mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$. One can check that these are one the inverse of the others. □

For example, let $X \xrightarrow{f} Y$ be a morphism of algebraic varieties (or any example in point (1) from 2.1.1); then for any open $V \subset Y$ and any regular function $h : V \to K$ the composition $f^{-1}(V) \xrightarrow{f} V \xrightarrow{g} K$ is a regular function on $f^{-1}(V) \subset X$, and so we obtain a morphism of sheaves of rings $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$.

Remark 2.23. Let X, Y be differentiable (or complex) manifolds, and let $f : X \to Y$ be a continuous morphism. Then we get a morphism as before $f^{\#} : \mathscr{C}_Y \to f_*\mathscr{C}_X$. Since $\mathscr{C}_X^\infty \subset \mathscr{C}_X$ is a subsheaf and f_* is left-exact, also $f_*\mathscr{C}_X^\infty \subset f_*\mathscr{C}_X$ (similarly, $f_*\mathscr{O}_X^{\text{hol}} \subset f_*\mathscr{C}_X$)). Show that f is differentiable resp. holomorphic if and only if $f^{\#}(\mathscr{C}_Y^\infty) \subset f_*(\mathscr{C}_X^\infty)$ resp. $f^{\#}(\mathscr{O}_Y^{\text{hol}}) \subset f_*(\mathscr{O}_X^{\text{hol}})$.

This means that the sheaves $\mathscr{C}_X^\infty$ or $\mathscr{O}_X^{\text{hol}}$ carry all the information that turn X into a differentialble or complex manifold. For example, we can substitute the (cumberstone) notion of atlas with the one of sheaf.

Definition 2.24. A ringed space $(X, \mathscr{O}_X)$ is a topological space X together with a sheaf of rings. We say that $(X, \mathscr{O}_X)$ is a locally ringed space if the stalk of $\mathscr{O}_X$ at any point $x \in X$ is a local ring.

A morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of locally ringed spaces is a pair $(f, f^{\#})$ where $f : X \to Y$ is a continuous map and

$$f^{\#} \in \operatorname{Hom}_X(f^{-1}\mathscr{O}_Y, \mathscr{O}_X) = \operatorname{Hom}_Y(\mathscr{O}_Y, f_*\mathscr{O}_X)$$

is a morphism of sheaves of rings satisfying: for every $x \in X$, the induced map on stalks

$$\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,y}$$

is a morphism of local rings (meaning that sends the maximal ideal to the maximal ideal).

For example, for any map of rings $A \to B$ and any prime $\mathfrak{p} \subset B$ the induced map $A_{\mathfrak{p}^c} \to B_{\mathfrak{p}}$ is local by the definition of the contraction $\mathfrak{p}^c$. This shows that morphisms of algebraic varieties are automatically local. On the other hand, note that the inclusion of $K[[t]]$ in its fraction field $K((T))$ is not a map of local rings.

Definition 2.25. An algebraic variety over K is a locally ringed space $(X, \mathscr{O}_X)$ which is covered by open subsets U_i such that $(U_i, \mathscr{O}_{|U_i})$ is isomorphic, as a locally ringed space, to an affine K-variety (here, $\mathscr{F}_{|U}$ denotes the inverse image of $\mathscr{F}$ along the open immersion $U \subset X$).

For example, let $(X, \mathscr{O}_X)$ be an affine algebraic variety and let $U \subset X$ be an open subset. Then $(U, \mathscr{O}_{|U})$ is an algebraic variety. It is called a *quasi-affine variety.*

Remark 2.26. Quasi-affine varieties need not be affine. The classical example is $U = K^2 \setminus 0$ with the induced sheaf of functions. A regular function $U \to K$ is in particular a rational function $h = f/g \in K(x, y)$ with $f, g \in K[x, y]$ coprime. Assume that g is not a unit. We claim that $V(g) \not\subset V(f)$ inside K^2. Let $\prod_i \pi_i^{\alpha_i}$ be a prime decomposition of f and $\prod_j \pi_j'^{\alpha_i'}$ be one of g. Then $\pi_i \neq \pi_j'$ for any i, j. Suppose $V(g) \subset V(f)$ and fix a prime π dividing g. Then $V(\pi)$ is an irreducible variety of dimension one. Since $V(\pi) \subset \bigcup_j V(\pi_j')$ and each $V(\pi_j')$ is irreducible of dimension one, we have that $V(\pi) = V(\pi')$ for some π' dividing f. But this means that $\pi = \pi'$ up to units by Nullstellensatz, which is a contradiction to coprimality. So $V(\pi) \neq V(\pi_j')$ for every j. Now $V(\pi_j') \cap V(\pi)$ is closed of $V(\pi)$, hence it must be a bunch of points (for it cannot be the whole thing). Hence up to finitely many points none of the points in $V(\pi)$ are in $V(f)$. Since $V(\pi)$ contains infinitely many points we conclude that there must be some $u \in K^2 \setminus 0$ such that $g(u) = 0$ but $f(u) \neq 0$, showing that f/g is not regular at u.

Another proof: assume that $h \in \cap_{\mathfrak{m} \neq (x_0, x_1, x_2)} K[x, y]_{\mathfrak{m}} \subset K(x, y)$. Using the same trick as in point (3) of Theorem 2.10 we see that $I = \{g \in K[x, y] : \ gh \in K[x, y]\}$ is an ideal such that $\sqrt{I} = (x_0, x_1, x_2)$. Hence $(x_0, x_1, x_2)^k \subset I$ for some $k > 0$ because (x_0, x_1, x_2) is maximal. This means that $x_i^k h \in K[x, y]$ for every i. Now use that $K[x, y]$ is a UFD and find the contradiction as before.

We will show later in the course (but you can try to prove it yourself now) that for affine varieties $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ any morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is induced by a unique morphism of varieties, hence by a unique map of K-algebras $A(Y) = \Gamma(Y, \mathcal{O}_Y) \to A(X) = \Gamma(Y, \mathcal{O}_X)$.

3 Lecture III / IV

The projective space $\mathbb{P}^n(K)$ is the simplest and most useful compactification of the affine spaces K^n. Originally, it was discovered by painters to study prospective and its use in algebraic geometry was initiated much later, that is, when the need to work with complete (or proper or 'compact') varieties became apparent. We prefer to work with such spaces for many reasons, most notably: finitely generation of cohomology, existence of intersection theory e.g., Bézout's theorem, Poincaré duality, existence of limits, etc).

The projective plane has its origins in practical problems, such as the mathematical understanding of perspective, and this motivates its abstract definition in a precise way which is not so hard to illustrate.

Consider the (x, y)-plane $P \subset \mathbb{R}^3$, with an observer standing upright on it: their feet are at the origin $O = (0, 0, 0)$ and their head is at the point $H = (0, 0, 1)$ (up to rescaling). On the whole plane P there is a picture which the observer views from H. As the observer looks farther across P, the picture appears increasingly compressed toward the horizon and when its line of sight becomes parallel to P, then the observer is looking directly at the *horizon* and raising the gaze further above only reveals empty space. Thus, although P is infinite, from the point of view of the observer it appears to bounded by the horizon. For example consider an infinite railway drawn on P, with O at its midpoint. Looking straight down at your feet the two rails appear as parallel lines. As you lift your gaze along the direction of the railway, the rails seem to converge until they appear to meet at the horizon. The horizon, however, is not a physical object but rather a visual effect, so nothing can really meet there. In projective geometry we incorporate the horizon in a mathematical way to P and thus obtain a new space, the *projective plane.* In this space the two lines will then really intersect on the horizon, also called the *line at infinity.*

This also gives an example of a geometry where Euclid's fifth postulate does not hold.

3.1 Projective spaces

Let K be as usual an algebraically closed field.

Definition 3.1. The n-th dimensional projective space is defined as

$$\mathbb{P}^n(K) := (K^{n+1} \setminus 0)/K^\times$$

where $K^\times$ acts on $K^{n+1} \setminus 0$ via

$$\lambda \cdot (x_0, \dots, x_n) \mapsto (\lambda x_0, \dots, \lambda x_n).$$

One denotes $[x_0 : \dots : x_n]$ the point corresponding to $(x_0, \dots, x_n) \in K^{n+1} \setminus 0$. Similalrly, if W is any K-vector space we let $\mathbb{P}(W) = (W \setminus 0)/K^\times$.

Thus $\mathbb{P}(W)$ is in natural bijection with the set of lines $L \subset W$ passing through the origin (which should be interpreted as the point H if we keep the analogy from the introduction, i.e., each line corresponds to a point in your view).

Remark 3.2. If $K = \mathbb{C}$ and $\mathbb{D}^n = \{\mathbf{z} \in \mathbb{C}^{n+1} : |\mathbf{z}| = 1\}$ then the induced map $\mathbb{D}^n \to \mathbb{P}^n(\mathbb{C})$ is surjective, which shows in particular that $\mathbb{P}^n(\mathbb{C})$ is compact when equipped with the analytic topology.

Once we define properness for schemes, we will prove that $\mathbb{P}^n$ is compact by purely algebraic methods.

3.1.1 Zariski topology on $\mathbb{P}^n(K)$

Consider $K^{n+1} \setminus 0 \subset K^{n+1}$ with the induced Zariski topology, and denote by $\pi : K^{n+1} \setminus 0 \to \mathbb{P}^n(K)$ the quotient map. We endow $\mathbb{P}^n(K)$ with the quotient topology, so that $Z \subset \mathbb{P}^n(K)$ is closed if and only if $\pi^{-1}(Z) \subset K^{n+1} \setminus 0$ is closed. This means that there is a unique Zariski closed $C(Z) \subset K^{n+1}$ such that $C(Z) \setminus 0 = \pi^{-1}(Z)$. So $C(Z) = V(I)$ for some ideal $I \subset K[x_0, \cdots, x_n]$. By construction, if $x \in C(Z)$ and $k \in K$ then $k \cdot x \in C(Z)$ too: that is, if $x \in C(Z) \setminus 0$ then also the line L_x joining x to 0 is contained in $C(Z)$. Such closed subsets are called cones (with vertex 0 and base Z). So what kind of ideals $I \subset K[x_0, \cdots, x_n]$ yield cones as vanishing locus? One can figure out the answer easily 'by hand'. More conceptually, for any $k \in K^\times$ consider the automorphism of rings $\lambda_k : K[x_0, \cdots, x_n] \to K[x_0, \cdots, x_n]$ sending $x_i \mapsto kx_i$. This defines an action of $K^\times$ on $K[x_0, \cdots, x_n]$. So $V(I)$ is a cone if and only if $\lambda_k(I) = I$ for every $k \in K^\times$. We decompose now the polynomial ring into eigenspaces for this action:

$$K[x_0, \cdots, x_n] = \bigoplus_{d \geq 0} K[x_0, \cdots, x_n]_d$$

where $K[x_0, \cdots, x_n]_d$ consists of the K-vector space of homogeneous polynomials of degree d, i.e.,

$$K[x_0, \cdots, x_n]_d = \{P \in K[x_0, \cdots, x_n] : \lambda_k(P) = k^d P \text{ for every } k \in K^\times\}.$$

Proposition 3.3. *An ideal $I \subset K[x_0, \cdots, x_n]$ is invariant for the action of $K^\times$ if and only if it is generated by homogeneous polynomials.*

Such ideals are called *homogeneous* ideals.

Proof. If I is generated by homogenous polynomials, then it is clearly invariant under the action of $K^\times$. Now, assume that I is invariant, pick $f \in I$ and decompose $f = \sum_d f_d$ into homogeneous components. We claim that each $f_d \in I$ as well. We prove this by induction on the number $N = |\{d : f_d \neq 0\}|$. So if $N = 1$ there is nothing to prove. If $N > 1$ let d_0 be the maximum of $\{d : f_d \neq 0\}$ and let $\mu \in K$ be a primitive d_0-root of unity. Then $f - \delta_{\mathfrak{m}} u(f) \in I$ by assumption and

$$f - \lambda_{\mathfrak{m}} u(f) = \sum_{d<d_0} (1 - \mu^d) f_d$$

Since μ is primitive $1 - \mu^d \neq 0$ for any $0 < d < d_0$ hence by induction each $f_d \in I$ for $d < d_0$ and so also $f_d \in I$. But this implies that the ideal I is generated by the homogeneous components of its elements, which means that I a homogeneous ideal. □

Thus every homogeneous ideal gives a closed subset of $\mathbb{P}^n(K)$, with one small caveat. Note that the action of $K^\times$ extends naturally to an action on K^{n+1} and $K^{n+1}/K^\times = \{0\} \sqcup \mathbb{P}^n(K)$. The homogeneous ideal $(x_0, \dots, x_n)$ then would correspond to the point $\{0\}$ of $K^{n+1}/K^\times$. Since we are throwing this point away, we should consider only homogeneous ideals which do not contain the ideal $(x_0, \dots, x_n)$, which is called the *irrelevant ideal.* Note in particular that $V(x_0, \cdots, x_n) = \emptyset$. One can prove in this way a correspondence analogue to the one from affine varieties:

Proposition 3.4. *Let $X = V(\mathfrak{p})$ be a projective variety and let $R = K[x_0, \dots, x_n]/\mathfrak{p}$ which is a graded ring in a natural way. Let R_+ be the ideal generated by homogeneous elements of positive degree. Then there is a one-to-one correspondence between closed subsets of X and radical homogeneous ideals of R strictly contained in R_+.*

Example 3.5. Take $F \in K[x_0, \cdots, x_n]_d$ for $d \geq 1$, and consider the homogeneous ideal $I = (F)$. Note that $F(kx_0, \cdots, kx_n) = k^d F(x_0, \cdots, x_n)$ hence the set

$$\{[x_0 : \ \dots, \ : \ x_n] \in \mathbb{P}^n(K) : \ F(x_0, \cdots, x_n) = 0\}$$

is a well-defined projective hypersurface.

An irreducible closed subset of $\mathbb{P}^n(K)$ is called a projective variety.

Natural open cover To describe the geometry of $\mathbb{P}^n(K)$ and of projective varieties in general, we turn the introduction into a mathematical construction. Consider any hyperplane $W \subset K^{n+1}$ (i.e. a linear subspace of dimension n) not passing through the origin. Every point $w \in W$ defines a unique line

$L_w \subset K^{n+1}$ passing through the origin and w, hence a unique point $\phi_W(w) \in \mathbb{P}^n(K)$. If $\overline{W}$ denotes the hyperplane parallel to W and passing through 0 (the directions at the horizon) then the only points of $\mathbb{P}^n(K)$ which are not in the image of ϕ_W are those lines inside $\overline{W}$, that is

$$\mathbb{P}^n(K) \setminus \operatorname{Im}(\phi_W) = \mathbb{P}(\overline{W}) \cong \mathbb{P}^{n-1}(K).$$

This means that the horizon is in fact a projective space of dimension one smaller. By repeating this yields decompositions

$$\mathbb{P}^1(K) \cong K \sqcup \{*\}$$

$$\mathbb{P}^2(K) \cong K^2 \sqcup \mathbb{P}^1(K) \cong K^2 \sqcup K \sqcup \{*\}$$

and so on. So in the projective line, we add only one point to K to obtain $\mathbb{P}^1(K)$, which is the infinity. In the projective plane we need to add all the directions parallel to the affine plane, which corresponds to the intuitive idea of horizon.

Proposition 3.6. *Let $W_0, \cdots, W_n \subset K^{n+1} \setminus 0$ be linearly independent hyperplanes $\overline{W}_0 \cap \cdots \cap \overline{W}_n = 0$. Then*

$$\bigcup \operatorname{Im}(\phi_{W_i}) = \mathbb{P}^n(K).$$

So every point of $\mathbb{P}^n(K)$ lies in the image of some ϕ_W; moreover, the map $\phi_W : W \to \mathbb{P}^n(K)$ will soon turn out to be an open immersion. In this way we get a local description of $\mathbb{P}^n(K)$ around every point: $\mathbb{P}^n(K)$ is covered by open subsets which are also affine algebraic varieties (in our case, K^n). In particular, it is an algebraic variety.

3.2 Functions on $\mathbb{P}^n_K$

If G is a group acting on some space X and $Y = X/G$ exists, then we expect a function on Y to be a function on X which is invariant under the action of G. Since for $n \geq 1$ the regular functions on $K^{n+1} \setminus 0$ are the same as the regular functions on K^{n+1} (why?) a regular function $\mathbb{P}^n(K) \to K$ should then correspond to a polynomial in $K[x_0, \dots, x_n]$ which is invariant under the action of $K^\times$, i.e., a constant polynomial. This heuristic argument shows an important feature of projective (or complete) varieties: any morphism to an affine variety must be constant.

On the other hand, let F be a homogeneous polynomial, and let $D_+(F) \subset \mathbb{P}^n(K)$ be the corresponding principal open subset. Now $\pi^{-1}(D_+(F)) = D_F \setminus 0 \subset K^{n+1} \setminus 0$ and $\mathcal{O}(D_F) = K[x_0, \dots, x_n]_F$. Using a similar reasoning, the regular functions $D_+(F) \to K$ should correspond to the fractions in $K[x_0, \dots, x_n]_F$

which are invariant under the action of $K^\times$. Note that we can extend the degree function on the whole $\mathrm{Frac}(K[x_0,\dots,x_n])$, so the invariant functions correspond to the degree zero part of $K[x_0,\dots,x_n]_F$, which is a subring denoted by $K[x_0,\dots,x_n]_{(F)}$. Then every element $G/F^k \in K[x_0,\dots,x_n]_{(F)}$ where G is homogeneous of degree $k\deg(F)$ describes a regular function on $D_+(F)$ via the rule

$$D_+(F) \ni [x_0 : \ \dots \ : x_n] \mapsto \mathrm{Frac}G(x_0,\dots,x_n)F(x_0,\dots,x_n)^k$$

which is indeed a well-defined function.

Now, note that the open subsets $D_+(F)$ form a basis for the Zariski topology, more precisely: any open subset is the finite union of prinicipal open subsets (and any finite intersection of principal open subsets is again principal).

Excercise 3.7. Let X be a topological space and let $\{U_i\}_{i\in I}$ be a basis for the topology for X, as above. Assume that for any $i \in I$ we have a group $\tilde{\mathscr{F}}(U_i)$ together with a compatible system of restriction maps. Show that the same construction of the sheafification yields a unique sheaf $\mathscr{F}$ on X.

In this way, we can define the sheaf of rings $\mathcal{O}_{\mathbb{P}^n}$. Similarly, if $\mathfrak{p} \subset K[x_0,\dots,x_n]$ is a homogeneous prime ideal and $R = K[x_0,\dots,x_n]/\mathfrak{p}$ is the associated graded algebra, we define the sheaf $\mathcal{O}_X$ on $X = V(\mathfrak{p})$ by declaring $\mathcal{O}_X(D_+(F)) = R_{(F)}$ for every $F \in R$. The pair $(X, \mathcal{O}_X)$ is then called a projective variety.

3.2.1 Projective varieties are locally affine algebraic varieties

Let now $\mathfrak{p} \subset K[x_0,\cdots,x_n]$ be a homogeneous ideal and let $X = V(\mathfrak{p})$ be the associated projective variety. Let $R = K[x_0,\cdots,x_n]/\mathfrak{p}$ which is a graded ring. Fix a homogeneous element $F \in R$ (of some degree $d \geq 1$) and let $R_{(F)}$ denote the degree–zero part of the graded ring R_F. We will prove that the open subset $D_+(F) := X \setminus V(F) = \{x \in X \mid F(x) \neq 0\}$ is (functorially) isomorphic to the set of maximal ideals of $R_{(F)}$:

$$D_+(F) \cong \ \mathrm{mSpec}\left(R_{(F)}\right).$$

Note that $R_{(F)}$ is a finitely generated K-algebra, so $\mathrm{mSpec}\left(R_{(F)}\right)$ is an affine algebraic varieties.

- If $x = [x_0,\dots,x_n] \in D_+(F)$ then $F(x) \neq 0$ and so we have a well-defined evaluation map $R_F \to K$ evaluating each rational function at $(x_0,\dots,x_n)$. We restrict this map to the degree zero part and obtain $\mathrm{ev}_x : R_{(F)} \to K$; notice that this is well-defined (i.e., does not depend on the chosen representative of $(x_0,\dots,x_n)$ of x). The kernel of ev_x is then a maximal ideal $\mathrm{m}_x \in \mathrm{mSpec}\left(R_{(F)}\right)$, which defines the map $D_+(F) \to \mathrm{mSpec}(R_{(F)})$.

- Let now $\mathfrak{m} \in \operatorname{mSpec}\left(R_{(F)}\right)$ be a maximal ideal. The extension $\mathfrak{m}^e$ of $\mathfrak{m}$ to R_F is then $\bigoplus_{d\in\mathbb{Z}} F^d\mathfrak{m}$. Choose any $\lambda \in K^\times$ and consider the unique ring map $R_F \to K$ sending $R_{(F)}$ to $R_{(F)}/\mathfrak{m} = K$ and F to λ. This defines a maximal ideal of R_F hence a point $(x_0, \dots, x_n) \in K^{n+1}$. This cannot be 0 because $F(x_0, \dots, x_n) = \lambda \neq 0$ and therefore defines a point $[x_0 : \dots : x_n] \in D_+(F)$. Different choices of λ yield the same point.

By contruction, these are one the inverse of the others. As we shall prove later in more generality the map can be upgraded to an isomorphism of ringed spaces $(D_+(F), \mathcal{O}_{|D_+(F)}) \cong (\operatorname{mSpec}(R_{(F)}), \mathcal{O}_{\operatorname{mSpec}(R_{(F)})})$. In particular:

Corollary 3.8. *Every principal open subset of a projective variety is an affine variety in a natural way.*

Example 3.9.

- Let $W \subset K^{n+1}$ and $\overline{W}$ be as before. Then $\overline{W}$ is the zero locus of a homogeneous degree one polynomial F. Let us compute $K[x_0, \cdots, x_n]_{(F)}$. We can assume that $F = x_i$ for some i; then $K[x_0, \cdots, x_n]_{(x_i)} = K[x_0/x_i, \dots, x_n/x_i]$ which is the polynomial ring in the n-variables x_j/x_i for $j \neq i$. Show that $W \cong \operatorname{mSpec}(K[x_0, \dots, x_n]_{(F)})$ in a natural way and that the $\phi_W : W \to \mathbb{P}^n(K)$ corresponds to the identification

  $$\operatorname{mSpec}(K[x_0, \cdots, x_n]_{(F)}) \xrightarrow{\sim} D_+(F)$$

  from before.

- Let $W_i \subset K^{n+1}$ be the hyperplanes $V(x_i - 1)$ so that $\overline{W}_i = V(x_i)$. Let $X \subset \mathbb{P}^n(K)$ be a subvariety. To understand X we shall 'project it' to the hyperplanes W_i: Let $D_+(x) \cap X$ be the principal open subset corresponding to the restriction of x_i to X. If this is empty, then $X \subset V(x_i) \cong \mathbb{P}^{n-1}$ so we can assume that $D_+(x_{i|X}) \neq \emptyset$ up to moving to a smaller ambient space. The restriction of $D_+(x_i) \xrightarrow{\cong} W_i$ to X then identifies $D_+(x) \cap X$ with an algebraic variety $X_i \subset W_i$. If we understand all the algebraic varieties X_i then we can recover X by glueing them over their intersections, since the varieties $X_i \subset X$ form an open cover of X.

  For example, let $F \in K[x_0, \dots, x_n]$ be a homogeneous polynomial of degree d. Define $F_i = F/x_i^d \in K[x_0, \dots, x_n]_{(x_i)} \cong K[\dots x_j/x_i \dots]$. So F_i is a polynomial of degree d obtained by 'dehomogenizing' F at its i-th coordinate. If $X = V(F)$ then $X_i = V(F_i) \subset W_i \cong K^n$. This is very useful for concrete computations. If $X = V(\mathfrak{p})$ then to obtain X_i one dehomogenise the generators of $\mathfrak{p}$ at the i-th place and consider the ideals $\mathfrak{p}_i \subset K[\dots x_j/x_i \dots]$ generated by those.

3.2.2 Morphisms to affine varieties

We now prove the following:

Theorem 3.10. *Let $(X, \mathcal{O}_X)$ be a projective variety and let $(Y, \mathcal{O}_Y)$ be an affine variety. Then any morphism $f : X \to Y$ is constant.*

Proof. It is enough to show that any morphism $X \to K$ is constant, that is, it is enough to show that $\Gamma(X, \mathcal{O}_X) = K$. Let $X = V(\mathfrak{p})$ and let $R = K[x_0, \dots, x_n]/\mathfrak{p}$ be the corresponding graded ring. We denote by F the fraction field of R and by F_0 its degree-zero part. For every Zariski open $U \subset X$ we have a natural inclusion $\Gamma(U, \mathcal{O}_{|U}) \subset F_0$ and therefore we can treat all these rings as subrings of the same field. Now, for any $F \in R$ homogeneous of degree $d > 0$ we have $\Gamma(D_+(F), \mathcal{O}) = R_{(F)} \subset F_0$ and therefore $\Gamma(X, \mathcal{O}_X) = \bigcap_F R_{(F)}$. Now note that $x_1, \cdots, x_n$ generate R as a graded ring (we can throw away the $x_i \in K$). So for every $g \in \Gamma(X, \mathcal{O}_X)$ there must be $N > 0$ such that for $x_i^N g \in R_N$. Now we use a trick (see Hartshorne): replace N with $N(n+1)$ and consider a monomial in $x_1, \cdots, x_n$ of degree $N(n+1)$. Then at least one x_i must appear with power $\geq N$ and therefore $fR_{N(n+1)} \subset R_{N(n+1)}$. This also implies $f^k R_{N(n+1)} \subset R_{N(n+1)}$ for every $k \geq 0$. But then also $f^k R_M \subset R_M$ for every $M \geq N(n+1)$.

Consider now the graded ring $R[f] \subset F$. We have an inclusion of R-modules (forget the gradings)

$$R \subset R[f] \subset x_i^{-N} R \subset F$$

for some $x_i \neq 0$; but R is Noetherian and $x_i^{-N} R$ is obviously finitely generated R-module; so also the submodule $R[f]$ must be finitely generated, i.e., f is integral over R. If R were integrally closed (which means that the variety is normal) then we would conclude that $f \in R$ hence $f \in R_0 = K$. In general, write an equation $f^m + a_{m-1} f^{m-1} + \cdots + a_0 = 0$ with $a_i \in R$. Finally, considering the degree zero part of the equation we can assume $a_i \in K$ and, finally, $f \in K$ since K is algebraically closed. □

If we knew that X were proper (i.e. complete), then we could also prove the result as follows: let $f : X \to K$ by a function and consider the induced function $f : X \to K \subset \mathbb{P}^1(K)$. Since X is proper $f(X) \subset \mathbb{P}^1(K)$ is closed. But the only closed subsets of $\mathbb{P}^1(K)$ which are contained in the principal open subset K are the finite sets. Since X is connected, $f(X)$ must be a point.

Remark 3.11. In complex geometry this is a manifestation of the maximum modulus principle. That is, let X be a compact complex manifold and let $f : X \to \mathbb{C}$ be a holomorphic function. Since X is compact there must be a point $x \in X$ such that $|f(x)|$ achieve its maximum. Then take a small open subset $x \in U \subset X$. By the maximum module principle, $f_{|U}$ cannot achieve its maximum in U unless f is constant.

Let us show how the proof works in the easiest example. Consider $X = \mathbb{P}^1(K)$. Then we can cover X with the two open subsets $U_0, U_1 \cong K$. Now a function $f_0 : U_0 \to K$ is represented by a polynomial $P(x_1/x_0)$ and similarly a function $f_1 : U_1 \to K$ is represented by a polynomial $Q(x_0/x_1)$. In practice, this means that $f([x_0, x_1]) = P(x_1/x_0)$ if $x_0 \neq 0$ and $f([x_0, x_1]) = Q(x_0/x_1)$ if $x_1 \neq 0$. The compatibility condition simply means that $P(x_1/x_0) = Q(x_0/x_1)$ if both $x_0, x_1 \neq 0$. By putting $t = x_1/x_0$ we then have $P(t) = Q(t^{-1})$ which immediately shows that both P and Q are constant.

Corollary 3.12. *Let $Z \subset \mathbb{P}^n(K)$ be a positive dimensional projective variety and let F be homogeneous of degree $d \geq 1$. Then $V(F) \cap Z \neq \emptyset$.*

Proof. If $V(F) \cap Z = \emptyset$ then $Z \subset D_+(F)$ which is affine. Hence Z must be a point. □

Alert on graded rings We cannot expect the same dictionary between finitely generated K-algebras and affine varieties to work also for graded rings, and it is worthwhile to study the failure of this functoriality. For simplicity we assume that all graded rings are generated in degree one, as it happens for the graded quotients of $K[x_0, \dots, x_n]$. A morphism $f : R \to S$ of degree d is by definition a morphism of rings such that $f(R_n) \subset S_{nd}$. For example, $\phi : K[y_0, \dots, y_m] \to K[x_1, \dots, x_n]$ has degree d if and only if $f(y_i) = F_i$ is homogeneous of degree d for every i. We also define $R(d)$ as the graded ring $\oplus_n R_{nd}$ which comes with a natural inclusion $R(d) \subset R$ which has degree d. Note that $R(d)$ also is generated in degree one.

Now, one reason why such a morphism does not induce a map on the corresponding projective varieties is only due to the fact that certain maximal ideals may be sent to the irrelevant ideal. Geometrically, every such ϕ induces a map of affine cones $C(X) \to C(Y)$ and it may happen that some rays of $C(X)$ are collapsed to the origin, and so the correponding points of X cannot have a well-defined image. Now, in mSpec(S) and Spec(S) the ideals restricting to R_+ form a closed subset which corresponds to $V(R_+^e)$ due to the following lemma:

Lemma 3.13. *If $\phi : A \to B$ is a map of rings and I is an ideal of A then for a prime ideal $\mathfrak{p} \subset B$ we $I \subset \mathfrak{p}^c$ if and only if $I^e \subset \mathfrak{p}$.*

Proof. If $I \subset \mathfrak{p}^c$ then $I^e \subset (\mathfrak{p}^c)^e \subset \mathfrak{p}$. If $I^e \subset \mathfrak{p}$ then $I \subset (I^e)^c \subset \mathfrak{p}^c$. □

For example, if $F_1, \cdots F_m$ are homogeneous polynomials of degree d in n variables they should give an induced morphism $\mathbb{P}^n(K) \setminus V(F_1, \dots, F_m) \to \mathbb{P}^m(K)$. We make this a statement:

Proposition 3.14. *Any map $f : R \to S$ as before induces a morphism of algebraic varieties $U \to Y$ where $U = X \setminus V(R_+^e)$ with the induced sheaves of functions.*

On the other hand, the fact that $V(R^e_+) \neq \emptyset$ does not automatically imply that the map above does not extend to the whole X:

Example 3.15. Let $F, G \in K[x_0, x_1, x_2]$ be polynomials of degree d and consider the induced map ϕ given by $x = [x_0 : x_1 : x_2] \mapsto [F(x) : G(x)]$. This is not defined in the common zeros of F and G, which is non-empty by the previous corollary. Let $x \in V(F) \cap V(G)$ and let $L \subset \mathbb{P}^2$ be any line passing through x. Then the restriction $\phi_{|L-\{x\}}$ extends to a morphism $\phi_L : L \to \mathbb{P}^1(K)$ (see the next section or prove it). Consider now the set of lines $\mathscr{L}_x = \{L \subset \mathbb{P}^2(K) : x \in L\}$. Note that $\mathscr{L}_x \cong \mathbb{P}^1(K)$. Thus for every $L \in \mathscr{L}_x$ we can associate a value $\phi_{|L}(x) \in \mathbb{P}^1(X)$. Show that this defines a morphism $\mathscr{L}_x \to \mathbb{P}^1(K)$ and determine it explicitly.

It is more difficult to construct examples when L is replaced by a higher dimensional variety.

3.3 Compactification of affine varieties

Projective spaces allow us to compactify affine varieties very easily. This may lead to singular projective varieties (where the singularities happen at infinity if your original variety was non-singular) but it is easy to check its behaviour at infinity.

Let $V \cong K^n$ and let $X \subset V$ be an affine variety and consider the morphism $V \to K^n \oplus K$ sending $v \mapsto (1, v)$. The image of this is then the hyperplane $W_0 = (x_0 - 1)$ and so X can be seen as a Zariski closed subset of $W_0 \cong D_+(x_0) \subset \mathbb{P}^{\mathrm{n}}(\mathbb{K})$. Its Zariski closure $\overline{X} \subset \mathbb{P}^{\mathrm{n}}(\mathbb{K})$ is then the sought for compactification of X.

To understand this, let us again look at the case of hypersurfaces $X = V(f)$ where $f \in K[x_1, \cdots, x_n]$ has degree d. We let $F = x_0^d f(x_1/x_0, \cdots, x_n/x_0)$ This is now homogeneous and $\overline{X} = V(F)$. For example let $\mathbb{P}^{\mathrm{n}}(\mathbb{K}) \setminus D_+(x_0) = \mathbb{P}(W)$ be the hyperplane at infinity. Then if f_d is the homogeneous part of f of maximal degree we have

$$\overline{X} \cap (\mathbb{P}^{\mathrm{n}}(\mathbb{K}) \setminus D_+(x_0)) = V(f_d) \subset \mathbb{P}(W).$$

Let us use this equation in some examples:

Example 3.16.
1. (Lines) This is again a verification of the introduction: see $\mathbb{P}^2(K)$ as the compactification of K^2 and take two lines $V(ax + by + c)$ and $V(a'x + b'y + c')$. Then these lines meet the line at infinity, which is isomorphic to $\mathbb{P}(K^2) = \mathbb{P}^1(K)$, respectively at $[a : b]$ and $[a' : b']$. But these are the same points if and only if the lines are parallel.

2. (Conics) Recall that an affine conic $C \subset K^2$ is the zero set of an irreducible quadratic polynomial $q(x,y) = q_2(x,y) + q_1(x,y) + c$. If we complete the affine plane as before and let $\overline{C}$ be the closure of C in $\mathbb{P}^2(K)$, then the intersection of C at infinity is the zero locus in $\mathbb{P}^1(K)$ of $q_2(x,y)$. This can either consists of two different points or of one point with multiplicitly two (in the right affine chart, this is nothing but a quadratic equation in one variable). The frist case happens when C is an hyperbola and the second when the line at infinity is tangent to $\overline{C}$) i.e., C is a parabola. If $K = \mathbb{R}$ there is another possibility, namely, the points of $V(q_2)$ are not real (e.g. $q_2 = x^2 + y^2$). In this case the zero locus of f in $\mathbb{R}^2$ does not intersect the line at infinity and so C must be compact, i.e., an ellipse.

3. (Easy Bezout) Let $F \in K[x_0, x_1, x_2]$ be homogeneous of degree $d > 0$ and let $L \subset \mathbb{P}^2(K)$ be any line not contained in $V(F)$ (this is automatically if F does not contain a linear factor). Prove that $L \cap V(F)$ consists precisely of d-points when counted with multiplicity (reduce the statement to the fundamental theorem of algebra).

4. (Conics are lines) We now show that if $F \in K[x_0, x_1, x_2]$ is a homogeneous irreducible polynomial of degree 2 the associated conic $C = V(F) \subset \mathbb{P}^2(K)$ is actually isomorphic to $\mathbb{P}^1(K)$. Take any point $P \in C$ and take any line $L \subset \mathbb{P}^1(K)$ (i.e, the zero set of a linear homogeneous equation) with $P \notin L$. For any $c \in C \setminus P$ let L_c be the unique line passing through P and c. Since $P \notin L$ we have that $L \neq L_c$ hence $L \cap L_c$ consists precisely of one point $\phi(c)$. Show that this defines a map $C \setminus P \to L \cong \mathbb{P}^1(K)$ which extends to an isomorphism $C \cong \mathbb{P}^1(K)$ (construct the inverse; note that you can choose L in any nice position). Compare this with the conic examples from the previous lecture.

3.3.1 Glimpse of properness

Let X be a compact topological space and let

$$f : [0,1) \to X$$

be a continuous function. If f is well-behaved around 1 (e.g. if $f(a_n)$ is Cauchy for every $a_n \in [0,1)$ converging to 1) then the limit $\lim_{x \to 1} f(x)$ must exist in X. In algebraic geometry we do not have open intevervals, and the best we can do is to replace them with one dimensional open subsets. For instance, we can replace $[0,1)$ with $U = K \setminus 0$ and consider a morphism $f : U \to X$ where X is some projective variety. We then ask: what does it mean that f is well-behaved around the missing point 0, and if well-behaved, does the limit always exist? It turns out that any such morphism is automatically well-behaved (the reason for this is that rational functions are meromorphic functions with poles as singularities. For instance the function $(0, 1] \to \mathbb{S}$ sending $t \mapsto e^{2\pi i/t}$ would still be analytic but at 0 it has an essential singularities which prevents the limit to exist) and that one can always extend f uniquely to a morphism $K \to X$, i.e., the limit $\lim_{x\to 0} f(x)$ always exist when X is projective (or more generally, complete). Let us prove this in the simplest case:

Proposition 3.17. *Let $f : K \setminus 0 \to \mathbb{P}^1(K)$ be an morphism of algebraic varieties. Then f extends uniquely to a morphism $\tilde{f} : K \to \mathbb{P}^1(K)$.*

Proof. Intuitively, we can write $f(u) = [g(u) : h(u)]$ for some polynomials $g, t \in K[t]$ and every $u \in K \setminus 0$. Now if either $g(0) \neq 0$ or $f(0) \neq 0$ the point $f(0)$ is well-defined. If on the other hand $h(0) = g(0) = 0$ this means that both h, g are divisible by some power of t, we let t^N be the maximal power dividing both polynomials. Then $[g(u) \dots h(u)] = [u^N g_0(u) \dots u^M h_0(u)] = [g_0(u) \dots h_0(u)]$ and we can assume without loss of generality that $g_0(0) \neq 0$. Hence this is again well-defined at 0, which shows that the morphism extends.

Let us now make this rigorous and let us unpack the statement. Let t be the coordinate of K vanishing at 0 and let U_0, U_1 be the standard open cover of $\mathbb{P}^1(K)$. Let $f^{-1}(U_i) =: V_i \subset K \setminus 0$, which are an open subsets, and pick $\tilde{V} \subset K \setminus 0$ be a small open subset such that $\tilde{V} \subset V_0 \cap V_1$. Finally, let $V \subset K$ be the open subset $\tilde{V} \cup \{0\}$.

Then by the sheaf property it is enough to show that $f_{|\tilde{V}} : \tilde{V} \to \mathbb{P}^1(K)$ extends to V. But $f(\tilde{V}) \subset U_i$ for $i = 1, 2$ and we can see both as a morphism of affine algebraic varieties. It is enough then to show that either $f_0 : \tilde{V} \to U_0$ extends to $V \to U_0$ or that $f_1 : \tilde{V} \to U_1$ extends to $V \to U_1$.

Since $V = \mathrm{mSpec}(K[x]_{x\cdot P(x)})$ where $P(x) = (x - a_1) \cdots (x - a_k)$ for some $a_i \neq 0$ and $U_0 = \mathrm{mSpec}(K[x_1/x_0])$ we know that f_0 is determined by an algebra morphism $K[x_1/x_0] \to K[x]_{x\cdot P(x)}$ which sends x_1/x_0 to some $g(x) = x^n g_0(x)$ with $g_0(0) \neq 0$ and $n \in \mathbb{Z}$. This simply means that the map sends $v \in V$ to $[1 : v^n g_0(v)]$. Now, if $n \geq 0$ then $g(x) \in K[x]_{(x)}$ hence $g(x) \in \Gamma(V, \mathcal{O}_V)$ by Theorem 2.10, which means that it is the restriction of a unique morphism $V \to U_0$. If on the other hand $n < 0$ this means that $\tilde{V} \to U_0$ has a pole at 0. To extend it, let us look at it as a function $f_1 : \tilde{V} \to U_1$. But then f_1 must correspond to the algebra morphism $K[x_0/x_1] \to K[x]_{x\cdot P(x)}$ which sends x_0/x_1 to $g(x)^{-1} = x^{-n} g_0(x)^{-1}$ with still $g_0(0) \neq 0$. But now $g(x)^{-1} \in K[x]_{(x)}$ because $-n > 0$, hence it must extend as before (we are simply rewriting the map as $v \mapsto [v^{-n} g_0(v)^{-1} : 1]$ and sending 0 to $[0 : 1] = U_1 \setminus U_0$). □

Remark 3.18. It is fundamental that U has dimension one, otherwise the limit might not be unique anymore. For example, let $U = K^2 \setminus 0$ and consider the map $f : U \to \mathbb{P}^1(K)$ sending (x_0, x_1) to $[x_0 : x_1]$ which is well-defined. Then this cannot be extended to a map $K^2 \to \mathbb{P}^1(K)$. To prove this, assume that it does, and let $f(0)$ be the value of the extension at 0. But the restriction of f to any line $V(y - \alpha x) \setminus 0$ is the constant value $[1 : \alpha]$. So if the extension existed, all these values must also be equal to $f(0)$, which is absurd. There is however a natural construction to extend the map at zero: the blow-up.

4 Lecture V/VI

In this lecture we shall introduce schemes. Recall that if R is any commutative ring with unity we let $\operatorname{Spec}(R)$ be the set of its prime ideals, endowed with the Zariski topology. More precisely, all closed subsets are of the form

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) : I \subset \mathfrak{p}\}$$

for any ideal $I \subset R$. Recall also that if $f \in R$ then $D(f) = \operatorname{Spec}(R) \setminus V(f)$ is called a principal open subset; these subsets form a basis for the Zariski topology on $\operatorname{Spec}(R)$, and the natural map $R \to R_f$ induces a homeomorphism $\operatorname{Spec}(R_f) \cong D(f)$.

Our main result of this lecture is the following:

Theorem 4.1. *Given any R-module M there is a unique sheaf $\tilde{M}$ on* $\operatorname{Spec}(R)$ *such that*

$$\Gamma(D(f), \tilde{M}) \cong M_f = M \otimes_R R_f \qquad \text{for every } f \in R.$$

We follow the proof in the Stacks project as we find it the most conceptual. Note that if R is integral and $M = R$, then basically the same proof of Theorem 2.10 would work. We shall prove Theorem 4.1 in various steps.

Step 1: some basics on fundamental opens Consider the set $\mathscr{B} = \{D(f)\}_{f \in R}$. This forms a basis for the topology, and by quasi-compactness of $\operatorname{Spec}(R)$ we also know that every open subset is a finite union of principal open subsets and that $\mathscr{B}$ is closed under finite intersections.

Lemma 4.2. *We have $D(g) \subset D(f)$ if and only if f is invertible in R_g, if and only if $g^e \in (f)$ for some $e \geq 1$. In this case, for every R-module M there is a natural induced map $M_f \to M_g$, which is an isomorphism whenever $D(g) = D(f)$.*

Proof. In fact, $D(g) = \{\mathfrak{p} \subset R : g \notin \mathfrak{p}\} = \operatorname{Spec}(R_g)$. If $f \in R_g$ is not invertible there must be a maximal ideal $\mathfrak{m} \subset R_g$ such that $f \in \mathfrak{m}$, i.e., $\mathfrak{m} \in D(g)$ but $\mathfrak{m} \notin D(f)$, a contradiction. If f is invertible in R_g, we can write $1 = (f/1) \cdot (h/g^n)$, i.e. $g^k(g^n - fh) = 0$ in R for some k, which shows the claim. Finally, if $g^e = fh$ for some $h \in R$ and $g \notin \mathfrak{p}$ then clearly $f \notin \mathfrak{p}$ either, and so $D(g) \subset D(f)$.

For the other statements: since f is invertible in R_g, the universal property of localization yields a unique map $R_f \to R_g$. The map $M_f \to M_g$ follows either again from the universal property or by noting that $M_f = M \otimes_R R_f$ and using the previous map. □

Step 2: reinterpretation of sheaf conditions Let $\mathscr{F}$ be a presheaf of abelian groups on a topological space X. Let $U \subset X$ be open, and let $\{U_i\}_{i\in I}$ be an open cover of U; put $U_{ij} = U_i \cap U_j$.

Lemma 4.3. *The presheaf $\mathscr{F}$ is a sheaf if and only if for every such U and cover $\{U_i\}$ the sequence*

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathscr{F}(U_i) \xrightarrow{\beta} \prod_{i,j\in I} \mathscr{F}(U_{ij})$$

is exact, where $\alpha = \prod \rho_{UU_i}$ and

$$\beta(s)_{ij} = (s_i)_{|U_{ij}} - (s_j)_{|U_{ij}}.$$

Proof. Injectivity at the first term is axiom (II) for sheaves; exactness in the middle is axiom (I). □

Step 3: sheaves on a basis of opens Let X be a quasi-compact topological space and let $\mathscr{B}$ be a basis for the topology closed under finite intersections.

Definition 4.4. A sheaf of abelian groups $\tilde{\mathscr{F}}$ on $\mathscr{B}$ is the data of an abelian group $\tilde{\mathscr{F}}(U)$ for each $U \in \mathscr{B}$ such that for every $U \in \mathscr{B}$ and every *finite* open cover $U = \bigcup_{i=1}^n U_i$ with $U_i \in \mathscr{B}$, the sequence of Lemma 4.3 is exact.

Proposition 4.5. *In the situation above, every sheaf $\tilde{\mathscr{F}}$ on $\mathscr{B}$ extends uniquely to a sheaf $\mathscr{F}$ on X satisfying $\tilde{\mathscr{F}}(U) = \mathscr{F}(U)$ for all $U \in \mathscr{B}$.*

Proof. For any $x \in X$ define

$$\mathscr{F}_x = \varinjlim_{x\in U\in\mathscr{B}} \tilde{\mathscr{F}}(U),$$

and define the sheaf $\mathscr{F}$ by the usual sheafification-by-germs recipe: for $V \subset X$ open,

$$\mathscr{F}(V) := \left\{ (s_x)_{x\in V} \in \prod_{x\in V} \mathscr{F}_x \;\middle|\; \begin{array}{l} \text{for every } x \in V \text{ there exist } U \in \mathscr{B},\ x \in U \subset V, \\ \text{and } S \in \tilde{\mathscr{F}}(U) \text{ such that } s_y = S_y \text{ in } \mathscr{F}_y \text{ for all } y \in U \end{array} \right\}.$$

This is a sheaf, and by construction it restricts to $\tilde{\mathscr{F}}$ on $\mathscr{B}$. □

Moreover, note that the stalks of $\mathscr{F}$ and $\tilde{\mathscr{F}}$ agree.

Remark 4.6. Given two sheaves $\mathscr{F}, \mathscr{G}$ on X, to describe a morphism $f : \mathscr{F} \to \mathscr{G}$ it suffices to define it on the basis $\mathscr{B}$ compatibly.

Final step All in all, we only need to prove that for any R-module M the assignment $D(f) \mapsto M_f$ defines a sheaf on the basis of principal opens. So pick a finite covering $D(f) = \bigcup_{i=1}^{n} D(g_i)$; we need to prove that

$$0 \longrightarrow M_f \longrightarrow \bigoplus_i M_{g_i} \longrightarrow \bigoplus_{i,j} M_{g_i g_j}$$

is exact. Note that $g_1, \dots, g_n$ generate the unit ideal of R_f. It is enough to prove:

Proposition 4.7. *Let R be a ring, M an R-module, and $g_1, \dots, g_n \in R$ generate the unit ideal. Then*

$$0 \longrightarrow M \longrightarrow \bigoplus_i M_{g_i} \longrightarrow \bigoplus_{i,j} M_{g_i g_j}$$

is exact.

Proof. It suffices to check exactness after localizing at each maximal ideal $\mathfrak{m}$. Since the g_i generate the unit ideal, we can assume $g_1 \notin \mathfrak{m}$, i.e. g_1 is a unit in $R_\mathfrak{m}$. Then $(M_{g_1})_\mathfrak{m} = M_\mathfrak{m}$ and $(M_{g_1 g_i})_\mathfrak{m} = (M_\mathfrak{m})_{g_i}$. Thus we may assume $g_1 = 1$. The first map is then injective. For exactness in the middle, if $(m_i)_i \in \bigoplus_i M_{g_i}$ maps to zero, note $m_1 \in M$, and $m_j - m_1 = 0$ in M_{g_j} for all j, which proves the claim. □

This completes the proof of Theorem 4.1. If we pick $M = R$ we get a sheaf of rings on Spec(R), usually denoted by $\mathscr{O}_{\mathrm{Spec}(R)}$. This should be considered the sheaf of regular functions on Spec(R).

Proposition 4.8 (Stalks). *For any $x \in \mathrm{Spec}(R)$ and any sheaf $\tilde{M}$ on $\mathrm{Spec}(R)$ coming from an R-module M, we have*

$$\tilde{M}_x \cong M_\mathfrak{p}$$

where $\mathfrak{p}$ is the prime ideal corresponding to x. In particular, the stalk of the structure sheaf $\mathscr{O}_{\mathrm{Spec}(R)}$ is the local ring $R_\mathfrak{p}$.

Proof. Note that $\mathscr{B}_x := \{D(f) : x \in D(f)\}$ is a cofinal system of open neighborhoods. Hence

$$\tilde{M}_x = \varinjlim_{D(f) \in \mathscr{B}_x} \tilde{M}(D(f)) = \varinjlim_{D(f) \in \mathscr{B}_x} M_f.$$

Now $x \in D(f)$ iff $f \notin \mathfrak{p}$. Ordering $\{f \in R : f \notin \mathfrak{p}\}$ by $f \le g$ if $D(g) \subset D(f)$, we have transition maps $M_f \to M_g$, and

$$\varinjlim_{D(f) \in \mathscr{B}_x} M_f = \varinjlim_{f \notin \mathfrak{p}} M_f.$$

There are natural maps $M_f \to M_{\mathfrak{p}}$ for $f \notin \mathfrak{p}$, inducing $\varinjlim_{f \notin \mathfrak{p}} M_f \to M_{\mathfrak{p}}$. This is injective: if $m/1 \in M_f$ maps to 0 in $M_{\mathfrak{p}}$, then some $g \notin \mathfrak{p}$ satisfies $gm = 0$, so m maps to 0 in M_{gf}. Surjectivity is clear. □

Corollary 4.9. *For any ring R, the space* Spec(R) *with $\mathscr{O}_{\mathrm{Spec}(R)}$ is a locally ringed space.*

Definition 4.10. An *affine scheme* is a locally ringed space isomorphic to

$$(\mathrm{Spec}(R), \mathscr{O}_{\mathrm{Spec}(R)})$$

for some ring R. A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ which is locally isomorphic to an affine scheme, i.e. there is an open cover $X = \bigcup_i U_i$ such that each $(U_i, \mathscr{O}_{X|U_i})$ is affine.

Remark 4.11. If $(\mathrm{Spec}(R), \mathscr{O}_{\mathrm{Spec}(R)})$ is an affine scheme and $f \in R$, then we can still consider f as a function on Spec(R), with the caveat that the codomain varies: the value at $\mathfrak{p} \in \mathrm{Spec}(R)$ is the image of f in $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = \kappa(x)$, the residue field of the corresponding point x.

For example, if R is a finitely generated integral K-algebra with K algebraically closed, then for a maximal ideal $\mathfrak{m}$ corresponding to $x \in \mathrm{mSpec}(R)$ we recover $f(x)$ via $f \equiv f(x) \bmod \mathfrak{m}$. For K-algebras, such an f yields a function $\mathrm{mSpec}(R) \to K$. On the other hand, the value at the generic point $(0) \in \mathrm{Spec}(R)$ is f viewed in Frac(R).

Similarly, if $Z \subset \mathrm{mSpec}(R)$ is irreducible with prime $\mathfrak{p}$, then $f_{|Z}$ is a regular function on Z, i.e. an element of $R/\mathfrak{p}$, and the "value" at $\mathfrak{p}$ is that function seen in $\mathrm{Frac}(R/\mathfrak{p})$ (the function field of Z). In particular, $f(\mathfrak{p}) = 0$ iff f vanishes on Z.

4.1 Morphisms of schemes

A morphism of schemes is a morphism of locally ringed spaces.

Proposition 4.12 (Morphisms to affine schemes). *Let $(X, \mathscr{O}_X)$ be a locally ringed space and $Y = \mathrm{Spec}(R)$ an affine scheme. Then there is a natural bijection*

$$\mathrm{Hom}_{l.r.s}(X, Y) \cong \mathrm{Hom}_{\mathrm{rings}}(R, \Gamma(X, \mathscr{O}_X)).$$

Let $f \in \mathrm{Hom}(X, Y)$; then $f : X \to Y$ is continuous and $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$. The map $f^\#$ induces $\psi_f : R \to \Gamma(X, \mathcal{O}_X)$, and $f \mapsto \psi_f$ is the bijection above.

Lemma 4.13. *For any $x \in X$ consider the composition*

$$R \xrightarrow{\psi_f} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$$

and let $\mathfrak{p} \subset R$ be the inverse image of $\mathfrak{m}_x$. Then $f(x) = \mathfrak{p}$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\Gamma(X, \mathcal{O}_X) \arrow[r] & \mathcal{O}_{X,x} \\
R \arrow[u, "\psi_f"] \arrow[r] & R_{\mathfrak{p}'} \arrow[u]
\end{tikzcd}$$

where $\mathfrak{p}' = f(x)$. The right vertical map is local, so the preimage of $\mathfrak{m}_x$ is $\mathfrak{p}'$. Commutativity gives $\mathfrak{p}' = \mathfrak{p}$. □

Lemma 4.14 (Generalization of principal open subsets). *Let $(X, \mathcal{O}_X)$ be a locally ringed space and $f \in \Gamma(X, \mathcal{O}_X)$. Then $D(f) = \{x \in X : f_x \notin \mathfrak{m}_x\}$ is open, and $f_{|D(f)}$ is invertible.*

Proof. Fix $x \in D(f)$. Since $f_x \notin \mathfrak{m}_x$, choose $g_x \in \mathcal{O}_{X,x}$ with $f_x g_x = 1$. For $x \in U$ small, represent g_x by $g \in \mathcal{O}_X(U)$. Then $gf - 1 \in \mathcal{O}_X(U)$ vanishes at x, hence vanishes on some open $V \subset U$; thus $V \subset D(f)$, and $f_{|V}$ is invertible. □

Proof of Proposition 4.12. We construct the inverse to $f \mapsto \psi_f$. Given $\psi \in \mathrm{Hom}(R, \Gamma(X, \mathcal{O}_X))$, define $f : X \to \mathrm{Spec}(R)$ by $x \mapsto \psi^{-1}(\mathfrak{m}_x)$. This is continuous since for $D(g) \subset \mathrm{Spec}(R)$ we have

$$f^{-1}(D(g)) = \{x : \psi(g) \notin \mathfrak{m}_x\} = D(\psi(g)).$$

To get $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$, it is enough to define on principal opens: for $D(g) \subset Y$ set

$$\mathcal{O}_Y(D(g)) = R_g \longrightarrow \mathcal{O}_X(D(\psi(g)))$$

using that $\psi(g)$ is invertible on $D(\psi(g))$. The induced stalk maps $R_\mathfrak{p} \to \mathcal{O}_{X,x}$ are local because $\mathfrak{p} = \psi^{-1}(\mathfrak{m}_x)$. □

Corollary 4.15. *The category of affine schemes is equivalent to the opposite category of rings.*

Relative point of view One core philosophy is relativity: instead of studying schemes over a fixed field, we study schemes over a base scheme.

Definition 4.16. Let S be a scheme. An *S-scheme* is a scheme X together with a morphism $X \to S$. A morphism of S-schemes is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
S \arrow[r, "\mathrm{Id}_S"] & S
\end{tikzcd}$$

Since $\mathbb{Z}$ is initial in rings, every scheme is naturally a scheme over $\mathrm{Spec}(\mathbb{Z})$. If R is a K-algebra, then $\mathrm{Spec}(R)$ is a $\mathrm{Spec}(K)$-scheme. Many properties are relative (properties of the morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(K)$), others are absolute.

In general, think of an S-scheme as a family of schemes parametrized by S. This will be clear once we introduce fibre products.

4.2 Sheaves of modules

If $(X, \mathcal{O}_X)$ is a ringed space, a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is a sheaf on X with $\mathcal{F}(U)$ an $\mathcal{O}_X(U)$-module for each open U, compatibly with restriction. For every ring R and R-module M we have the associated sheaf $\tilde{M}$ on $\mathrm{Spec}(R)$.

Proposition 4.17. *The association $M \mapsto \tilde{M}$ is exact.*

Proof. Exactness can be checked on stalks and follows from Proposition 4.8. $\square$

Every module M has a presentation $R^J \to R^I \to M \to 0$, which yields an exact sequence of sheaves

$$\mathcal{O}^J_{\mathrm{Spec}(R)} \longrightarrow \mathcal{O}^I_{\mathrm{Spec}(R)} \longrightarrow \tilde{M} \longrightarrow 0.$$

Definition 4.18 (Quasi-coherent sheaves). Let $(X, \mathcal{O}_X)$ be a locally ringed space. A quasi-coherent sheaf on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that for every $x \in X$ there exists an open $x \in U$ and a presentation

$$\mathcal{O}^J_{X|U} \longrightarrow \mathcal{O}^I_{X|U} \longrightarrow \mathcal{F}_{|U} \longrightarrow 0.$$

Thus, quasi-coherent sheaves are locally built from $\mathcal{O}_X$ using generators and relations.

Theorem 4.19. *Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. The following are equivalent:*

1. *$\mathscr{F}$ is quasi-coherent.*
2. *For every open affine $U = \operatorname{Spec}(R)$ and $f \in R$, the natural map*

   $$\Gamma(U, \mathscr{F})_f \longrightarrow \Gamma(D(f), \mathscr{F})$$

   is an isomorphism.
3. *For every open affine $U = \operatorname{Spec}(R)$ in X there exists an R-module M with $\mathscr{F}_{|U} \cong \tilde{M}$.*

Before the proof: in (2), since $\mathscr{F}$ is an $\mathscr{O}_X$-module, $\Gamma(D(f), \mathscr{F})$ is an R_f-module; by the universal property of localization we get the map $\Gamma(U, \mathscr{F})_f \to \Gamma(D(f), \mathscr{F})$.

Proof. To prove that (1) implies (2) we can assume that X itself is affine. Clearly then (2) holds if $\mathscr{F}$ is of the form $\tilde{M}$ since in this case $\Gamma(X, \mathscr{F})_f = M_f$. Now we can cover X with finitely many principal open subsets $D(g_i)$ such that $\mathscr{F}_i = \mathscr{F}_{|D(f_i)}$ has a presentation

$$\tilde{R}_i^J \to \tilde{R}_i^I \to \mathscr{F}_i \to 0$$

where we put $R_i = R_{g_i}$. But then if M_i is the R_i module defined by the same presentation we have $\mathscr{F}_i = \tilde{M}_i$ necessarily and so it satisfies (2). By the same reasoning, also $\mathscr{F}_{ij} = \mathscr{F}_{|D(g_i g_j)}$ satisfies (2). Now by the sheaf property we have an exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \prod_i \Gamma(D(g_i), \mathscr{F}) \longrightarrow \prod_{i,j} \Gamma(D(g_i g_j), \mathscr{F})$$

and since localization is exact we also get an exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F})_f \longrightarrow \prod_i \Gamma(D(g_i), \mathscr{F})_f \longrightarrow \prod_{i,j} \Gamma(D(g_i g_j), \mathscr{F})_f$$

and now we consider the map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \mathscr{F})_f \arrow[r] \arrow[d, "\alpha"] & \prod_i \Gamma(D(g_i), \mathscr{F})_f \arrow[r] \arrow[d, "\alpha'"] & \prod_{i,j} \Gamma(D(g_i g_j), \mathscr{F})_f \arrow[d, "\alpha''"] \\
0 \arrow[r] & \Gamma(D(f), \mathscr{F}) \arrow[r] & \prod_i \Gamma(D(f g_i), \mathscr{F}) \arrow[r] & \prod_{i,j} \Gamma(D(f g_i g_j), \mathscr{F})
\end{tikzcd}$$

but the last two vertical maps are isomorphisms by our discussion, and then also the first one is. So (2) holds.

Now we show that (2) implies (3). We can again assume that $X = U$ is affine. Put $M = \Gamma(X, \mathscr{F})$. Then we have

$$\Gamma(X, \mathscr{F})_f = M_f = \Gamma(D(f), \mathscr{F})$$

and so $\mathscr{F} = \tilde{M}$ by definition. Finally, the fact that (3) implies (1) is obvious. □

Corollary 4.20. *If $X = \operatorname{Spec}(R)$ is affine, the functor $M \mapsto \tilde{M}$ induces an equivalence between R-modules and quasi-coherent sheaves on X.*

Let now $\phi : R \to S$ be a map of rings and let M be an S-module. Viewing M as an R-module via ϕ, denote it by M'.

Proposition 4.21. *Let $f : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ be the induced map. Then $f_*(\tilde{M}) \cong \widetilde{M'}$.*

Proof. For $g \in R$ we have

$$\Gamma(D(g), f_*\tilde{M}) = \Gamma(D(\phi(g)), \tilde{M}) = M_{\phi(g)} = \Gamma(D(g), \widetilde{M'}).$$

□

4.3 Open and closed immersions

The two most basic morphisms of schemes (or locally ringed spaces in general) are open and closed immersions.

Definition 4.22. A map of locally ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is

1. an open immersion if $f : X \to Y$ is a homeomorphism onto an open subset and $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$ (intuitively, functions on opens of X are just functions on the same open of Y);
2. a closed immersion if $f : X \to Y$ is a homeomorphism onto a closed subset and $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective (intuitively, functions on X are locally restrictions of functions on Y).

For example, if $R \to R_f$ is a localization, then $\operatorname{Spec}(R_f) \to \operatorname{Spec}(R)$ is an open immersion onto $D(f)$. More generally, for a multiplicative set $S \subset R$, the induced map $\operatorname{Spec}(S^{-1}R) \to \operatorname{Spec}(R)$ identifies $\operatorname{Spec}(S^{-1}R)$ with $\bigcap_{s \in S} D(s)$. If this intersection is infinite, it need not be open; in particular, not every localization map yields an open immersion onto its image unless the intersection is open.

Closed immersions are more interesting, and nilpotents naturally appear:

Proposition 4.23. *Let R be a ring and $I \subset R$ an ideal. Then* $\operatorname{Spec}(R/I) \to \operatorname{Spec}(R)$ *is a closed immersion. Moreover, every closed immersion of affine schemes is of this form.*

Proof. Topologically, $\operatorname{Spec}(R/I)$ identifies with $V(I) \subset \operatorname{Spec}(R)$. On stalks, the induced map $R_{\mathfrak{p}} \to (R/I)_{\mathfrak{p}}$ is surjective for $\mathfrak{p} \supset I$, and is zero for $\mathfrak{p} \not\supset I$, hence $\mathcal{O}_R \to f_* \mathcal{O}_{\operatorname{Spec}(R/I)}$ is surjective. Conversely, if $\iota : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is a closed immersion, then $\mathcal{O}_{\operatorname{Spec}(R)} \to \iota_* \mathcal{O}_{\operatorname{Spec}(S)}$ is surjective; but $\iota_* \mathcal{O}_{\operatorname{Spec}(S)} = \tilde{S}$ where we see S as an R-module, and the associated map of rings $\phi : R \to S$ is surjective since all induced maps $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ are surjective (Atiyah–Macdonald, 3.9). □

Example 4.24. Let R be a ring, $\mathfrak{p} \subset R$ prime, $X = \operatorname{Spec}(R)$, $Y = \operatorname{Spec}(R/\mathfrak{p})$. The surjection $R \to R/\mathfrak{p}$ induces a closed immersion $Y = V(\mathfrak{p}) \subset X$. Also $R \to R/\mathfrak{p}^2$ induces a closed immersion $Y' = \operatorname{Spec}(R/\mathfrak{p}^2) \to X$, factoring $Y \to Y' \to X$. The kernel of $R/\mathfrak{p}^2 \to R/\mathfrak{p}$ is $\mathfrak{p}/\mathfrak{p}^2$, a natural quasi-coherent $\mathcal{O}_{Y'}$-module. Imagine that X is a $\mathscr{C}^\infty$-manifold and that Y is a submanifold. Let $\mathscr{I}_Y \subset \mathscr{C}^\infty_X$ be the ideal sheaf of functions vanishing along Y. Show that $\mathscr{I}_Y/\mathscr{I}_Y^2$ is naturally isomorphic to the normal bundle of Y in X. In this sense, nilpotens see infinitesimal information on how Y is embedded into X.

4.4 Proj and graded rings

An extremely useful way to construct non-affine schemes is the *Proj*-construction. Let $R = \bigoplus_{d \geq 0} R_d$ be a graded ring and $R_+ = \bigoplus_{d \geq 1} R_d$ the irrelevant ideal. An ideal $I \subset R$ is *homogeneous* if $I = \bigoplus_{d \geq 0} (I \cap R_d)$ (equivalently, if it is generated by homogeneous elements). Define

$$\operatorname{Proj}(R) = \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ homogeneous prime}, R_+ \not\subseteq \mathfrak{p}\}.$$

For homogeneous $f \in R$ of degree > 0, put

$$D_+(f) = \{\mathfrak{p} \in \operatorname{Proj}(R) : f \notin \mathfrak{p}\}.$$

These $D_+(f)$ are principal opens and form a basis; they cover $\operatorname{Proj}(R)$.

For $\mathfrak{p} \in \operatorname{Proj}(R)$, let $R_{\mathfrak{p}}$ be the localization of R at the multiplicative system of homogeneous elements not in $\mathfrak{p}$. This inherits a $\mathbb{Z}$-grading, and we let $R_{(\mathfrak{p})}$ be its degree-zero subring. For homogeneous f of positive degree, $R_{(f)}$ denotes the degree-zero subring of R_f.

We define a sheaf of rings $\mathcal{O}_{\operatorname{Proj}(R)}$ by declaring

$$\mathcal{O}_{\operatorname{Proj}(R)}(D_+(f)) := R_{(f)} \qquad \text{for homogeneous } f \text{ of deg } > 0,$$

with the obvious restriction maps.

Theorem 4.25 (Basic properties of Proj). *With the notation above:*

1. *The opens* $D_+(f)$ *cover* $\operatorname{Proj}(R)$.
2. *For every* $\mathfrak{p} \in \operatorname{Proj}(R)$*, the stalk* $\mathcal{O}_{\operatorname{Proj}(R),\mathfrak{p}}$ *is canonically isomorphic to the local ring* $R_{(\mathfrak{p})}$.
3. *There is a natural isomorphism of schemes*

$$\big(D_+(f), \mathcal{O}_{\operatorname{Proj}(R)}|_{D_+(f)}\big) \;\cong\; \big(\operatorname{Spec}(R_{(f)}), \mathcal{O}_{\operatorname{Spec}(R_{(f)})}\big).$$

In particular, $\operatorname{Proj}(R)$ *is a scheme.*

5 Letctures VII/ VIII

5.1 Some global properties of schemes

Schemes are very general objects, whereas varieties (which are in the end our object of interest) are very particular in respect, since they are obtained by glueing the Spec of finitely generated, integral K-algebras. So already to describe varieties among schemes we need a handful of properties.

Definition 5.1. Let X be a scheme. Then X is

1. *Connected* if the topological space of X is connected.
2. *Irreducible* if the topological space of X is irreducible;
3. *Reduced* if for every affine open $U = \mathrm{Spec}(A) \subset X$ we have that A has no nilpotents;
4. *Integral* if it is reduced and irreducible.

Let us characterise these properties:

Proposition 5.2. *Let X be a scheme, then*

1. *X is irreducible iff for every open affine* $\mathrm{Spec}(A) \subset X$ *the nilradical of A is prime;*
2. *X is reduced iff all its local rings have no nilpotents iff for every open affine* $\mathrm{Spec}(A) \subset X$ *the nilradical of A is zero;*
3. *X is integral iff for every open affine* $\mathrm{Spec}(A) \subset X$ *the ring A is a domain;*

Proof.

1. Suppose that X is irreducible and let $U \subset X$ be an open subset. Then also U is irreducible. But so if $U = \mathrm{Spec}(A)$ this means that A has a minimal prime ideal, which necessarily coincides with the nilradical. On the other hand, assume that for any such open the nilradical of A is not prime. Thus there are at least two different minimal primes in A, showing that $\mathrm{Spec}(A)$ cannot be irreducible.
2. Obvious, since nilpotent elements belong to every prime ideal, hence they inject into every localization.
3. Follows from the previous two.

□

Remark 5.3. Let X be an integral scheme. Then X has a unique generic point $\eta \in X$, i.e., a point such that $\bar{\eta} = X$. Let $U = \mathrm{Spec}(A) \subset X$ be any affine open. Since A is an integral domain $(0) \subset A$ is a prime ideal and η is just the image of (0) under the inclusion. This is easily well-defined. Its function field is $\mathrm{Frac}(A)$ which is independent on the chosen A.

Now, noetherianity and quasi-compactness.

Definition 5.4. A scheme X is quasi-compact if its topological space is quasi-compact. A scheme X is locally noetherian if we can cover it with affine opens $U_i = \mathrm{Spec}(A_i)$ where each A_i is a noetherian ring. It is noetherian if it is locally noetherian and quasi-compact (so we can find a finite number of such covers).

Note that many times in scheme theory we will encounter situations where a property is spelled out like 'there exists an affine open cover such that...'. In many situations – if not all – if this is true for one affine open cover then it is true for every affine open cover. Let us make an example:

Proposition 5.5. *Assume that X is locally noetherian. Then for every open affine $U = \mathrm{Spec}(A) \subset X$ we have that A is noetherian ring.*

Proof. We know that we can cover X with $U_i = \mathrm{Spec}(A_i)$ where each A_i is a notherian ring. So we can cover U with $U_i \cap U$. Since U is affine it is quasi-compact, hence we can find finitely many such opens. Moreover, for $a_i \in A_i$ note that A_{i,a_i} is also noetherian, and that we can cover each $U \cap U_i$ with finitely principal open of $\mathrm{Spec}(A_{i,a_i})$ each one noetherian. Thus we are reduced to prove the following: let $X = \mathrm{Spec}(A)$ be an affine scheme which can be covered by affine opens which are the Spec of noetherian rings. Then A is notherian.

Let $U = \mathrm{Spec}(B) \subset X$ be open with B noetherian. Then there must be some $f \in A$ such that $D(f) \subset U$ hence the inclusions $D(f) \subset U \subset X$ give maps of rings $A \to B \to A_f$. But then $B_f = A_f$ necessarily, hence also A_f is notherian.

Thus we are reduced to the following: let A be a ring and let $f_1, \cdots, f_n \in A$ such that $1 = (f_1, \cdots, f_n)$. Assume that each $A_i = A_{f_i}$ is notherian. Then A is notherian.

The proof rests now on this simple claim: in the situation above, let $\phi_i : A \to A_i$ be the localization map. Then for every ideal $I \subset A$ we have an equality

$$I = \bigcup_i \phi_i^{-1}(IA_i).$$

One inclusion is obvious. For the second, pick x in the intersection. Then for every i we can write $x = i_i / f_i^{n_i}$ for some $i_i \in I$. Hence we can find $m > n > 0$

such that for every i we have

$$f_i^m(i_i/f_i^n - x) = i_i f_i^{m-n} - f_i^m x = 0.$$

Now we can write $1 = \sum_i r_i f_i^m$ hence

$$x = (\sum_i r_i f_i^m)x = \sum_i r_i f_i^m x \in I$$

which proves the claim. Now, for every increasing sequence of ideals $I_1 \subset I_2 \subset \cdots$ we obtain for every i an increasing sequence $I_1 A_i \subset I_2 A_i \subset \cdots$ which must stabilize since A_i is notherian. Hence there is some $n >> 0$ such that $I_k A_i = I_{k+1} A_i$ for every $k \geq n$ and for every i. This implies that $I_k = I_{k+1}$ for every $k \geq n$ by the claim. □

In this way, we can say that Noetherianity is a local property of rings. In the previous lecture, we distinguished two types of properties for schemes: absolute properties (like the one above, which do not refer to a structure morphism) and relative properties, which pertain to morphisms rather than to schemes themselves. Before introducing the latter, we define one of the most important tools in algebraic geometry: fibre products.

5.2 Fibred products

Let $\mathscr{C}$ be any category and let

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

be a diagram of morphisms of $\mathscr{C}$.

Definition 5.6. The fibred product $X \times_S Y$, if it exists, is the universal object in $\mathscr{C}$ which makes the following diagram commute

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

By the universal property, if the fibred product exists, it is unique up to unique isomorphism.

Example 5.7. Let $\mathscr{C}$ be the category of sets. Then

1. If $S = \{*\}$ the fibred product is the usual product;

2. If $s \in S$ and $f : X \to S$ then $X \times_S \{s\} = X_s = f^{-1}(s)$.

3. If both $X, Y \subset S$ then $X \times_S Y = X \cap Y$.

Theorem 5.8. *Let X, Y be schemes over a base scheme S. Then the fibred product $X \times_S Y$ exists.*

Proof. The proof follows several steps.

- Step 1: all X, Y, S are affine. Write $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(C)$ so that we have maps of rings $C \to A$ and $C \to B$. We claim that $\operatorname{Spec}(A \otimes_C B)$ together with the natural maps $A, B \to A \otimes_C B$ is the fibred product $X \times_S Y$. We only need to check the universal property. So let Z be any scheme with maps $Z \to X, Y$ making the diagram commute. We know that this is equivalently given by ring maps $A, B \to \Gamma(Z, \mathcal{O}_Z)$ which restrict to the same map on C. Hence, by the universal property of tensor products, we can find a unique map of rings $A \otimes_C B \to \Gamma(Z, \mathcal{O}_Z)$ making the diagram commute

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & A \otimes_C B \arrow[dr, dashed, "\exists!"] & \\
 & & \Gamma(Z, \mathcal{O}_Z)
\end{tikzcd}$$

  This shows that $\operatorname{Spec}(A \otimes_C B)$ satisfies the universal property.

- Step 2: restrictions. If $X \times_S Y$ exists and $U \subset X$ is open, then $\pi_X^{-1}(U) \subset X \times_S Y$ is the fibred product $U \times_S Y$. This is easy using the universal property.

- Step 3: Glueing. Assume that $U_i \subset X$ is an open cover such that $U_i \times_S Y$ exists for every i. Then also $X \times_S Y$ exists. In fact, let $U_{ij} = U_i \cap U_j$. By the previous point $U_{ij} \times_S Y$ exists and it is in fact an open subscheme of $U_i \times_S Y$ as well as of $U_j \times_S Y$. So now we can glue the schemes $U_i \times_S Y$ along the open subschemes $U_{ij} \times_S Y$ to obtain a new scheme

$$\Big(\bigsqcup_i U_i \times_S Y \Big)_{/\sim}.$$

We need to show that this glued scheme is the required fibred product. So let $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$ be maps making the usual diagram commute and let $Z_i = f^{-1}(U_i)$. We get maps $f_i : Z_i \to U_i$ and $g_i : Z_i \to Y$ (the restriction of g to the open Z_i) making again the usual diagram commute. Hence, by the universal property for U_i, we get a unique map $Z_i \to U_i \times_S Y$ for every i. By the same universal property, these maps agree on the intersections $Z_{ij} = Z_i \cap Z_j$ in the sense that on such intersection they factor through the open immersions $U_{ij} \times_S Y \hookrightarrow U_i \times_S Y, U_j \times_S Y$ and match over the identifications. Hence they glue to a unique map $Z \to \left(\bigsqcup_i U_i \times_S Y\right)_{/\sim}$, which shows that the glued scheme represents the fibred product.

- Step 4: conclusion. All in all, this shows that $X \times_S Y$ exists whenever S is affine. Assume finally that S is not affine and cover it by affine opens $S_i \subset S$. Let X_i be the preimage of S_i in X and similarly for Y_i. Then $X_i \times_{S_i} Y_i$ exists for every i. It is easy to check that

  $$X_i \times_{S_i} Y_i \;\cong\; X_i \times_S Y,$$

  hence we use the previous point to conclude.

$\square$

Example 5.9.

- Fibres of a morphism. Let $f : X \to Y$ be a morphism and let $y \in Y$. Let $\kappa(y)$ be the residue field of y, so we have a natural map of schemes $\mathrm{Spec}(\kappa(y)) \to Y$. We define

  $$X_y \;=\; X \times_Y \mathrm{Spec}(\kappa(y))$$

  and we call it the fibre of X over y. For example, assume that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ and f is given by $A \to B$. For a prime $\mathfrak{p} \subset A$ corresponding to y we then have $X_y = \mathrm{Spec}\left(B \otimes_A \kappa(\mathfrak{p})\right)$. What are the primes of $B \otimes_A \kappa(\mathfrak{p})$? Consider the following commutative diagram

  $$\begin{tikzcd}
  B \arrow[r] \arrow[d] & A \arrow[d] \\
  B_{\mathfrak{p}} \arrow[r] \arrow[d] & A \otimes_B B_{\mathfrak{p}} \arrow[d] \\
  B_{\mathfrak{p}}/\mathfrak{p} = \kappa(\mathfrak{p}) \arrow[r] & A \otimes_B \kappa(\mathfrak{p})
  \end{tikzcd}$$

  Since the last horizontal map is necessarily an injection if the tensor is not zero, the commutativity of the diagram shows that the points in X_y

correspond precisely to the points of X mapped to y, i.e., to the prime $\mathfrak{q} \subset A$ such that $\mathfrak{q}^c = \mathfrak{p}$.

- Let R be any integral ring and consider $S = R[x_1, \cdots, x_n]/I$ where I is an ideal such that $I \cap R = (0)$. We thus get a morphism of schemes $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. Let $\mathfrak{p} \subset R$ be a prime ideal and let $I_{\mathfrak{p}}$ be the image of $I \otimes_R \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p})[x_1, \cdots, x_n]$ (note that this is not in general injective—see later flatness). Then for every prime $\mathfrak{p} \subset R$ the fibre is $\mathrm{Spec}\big(\kappa(\mathfrak{p})[x_1, \cdots, x_n]/I_{\mathfrak{p}}\big)$.

- Let now $R = K[t]$ and let $S = R[x, y]/(a(t)x + b(t)y - c)$ for some $a(t), b(t) \in R$ and $c \in K$. We can consider the natural map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. We can consider this as a family of lines. For any maximal ideal (assume K algebraically closed) corresponding to some $\alpha \in K$ the corresponding fibre is the line $K[x, y]/\big(a(\alpha)x + b(\alpha)y - c\big)$. For instance, note that if $c \neq 0$ and both $a(\alpha) = b(\alpha) = 0$ then the fibre is empty. It is otherwise a line passing through c;

- One can make many examples of the situation above. For example, $S = R[x, y]/(P_t(x, y))$ where $P_t(x, y) \in K[t, x, y]$. Then the fibre for the closed point $t = \alpha$ is nothing but $K[x, y]/\big(P_\alpha(x, y)\big)$. So we get a 'family of varieties' $t_0 \mapsto V(P(x, y, t_0))$.

- Base-extension: If $X \to S$ is an S-scheme and $S' \to S$ is a map of schemes, we call $X_{S'} = X \times_S S' \to S'$ the *base-extension* of X to S'. For example, let
$$X = \mathrm{Spec}\big(\mathbb{Q}[x, y]/(x^2 + 3y^2)\big).$$
Then X is an integral scheme. Let $\mathbb{Q} \subset \mathbb{Q}(\sqrt{-3})$ and consider the fibre product
$$X \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}\big(\mathbb{Q}(\sqrt{-3})\big) = \mathrm{Spec}\Big(\mathbb{Q}(\sqrt{-3})[x, y]/\big((x+\sqrt{-3}y)(x-\sqrt{-3}y)\big)\Big),$$
which is not irreducible anymore. In fact, properties like reduced/irreducible are not stable under base extension.

5.3 Some finiteness properties of morphisms

The most basic finiteness properties of morphisms are:

Definition 5.10. Let $f : X \to Y$ be a morphism of schemes. We say that f is

1. *Locally of finite type*: if there is an open cover $U_i = \mathrm{Spec}(A_i) \subset Y$ such that $f^{-1}(U_i) = V_i \subset X$ can be covered by affine opens $V_{i,j} = \mathrm{Spec}(B_{ij})$ with B_{ij} a finitely generated A_i-algebra for every i, j.

2. *Of finite type*: as above, and for each i we can choose finitely many $V_{i,j}$ (automatic if X is quasi-compact).
3. *Finite*: if there is an affine open cover $U_i = \operatorname{Spec}(A_i) \subset Y$ such that $f^{-1}(U_i)$ is affine, say $f^{-1}(U_i) = \operatorname{Spec}(B_i)$, and each B_i is a finitely generated A_i-module.

These properties are defined by the existence of an open cover of the base with the stated behavior, so they are *local on the base.* Closed immersions are easily seen to be finite morphisms. We have:

Proposition 5.11. *Let $f : X \to Y$ be as above.*

1. *If f is locally of finite type, then for* every *affine open $U = \operatorname{Spec}(A) \subset Y$ and* every *affine open* $\operatorname{Spec}(B) \subset f^{-1}(U)$*, the ring B is a finitely generated A-algebra.*
2. *If f is finite, then for* every *affine open $U \subset Y$ the preimage $f^{-1}(U)$ is affine, and the induced map of rings is integral.*

In general, if $f : X \to Y$ is a morphism and $U = \operatorname{Spec}(A) \subset Y$ is affine, there is no reason that $f^{-1}(U)$ is affine. When this happens, we say that f is *affine.* Thus finite morphisms are, in particular, affine.

To prove results of this kind, it is useful to have the following lemma.

Lemma 5.12. *Let A, B be rings, and suppose there exists a scheme U together with open immersions $U \hookrightarrow \operatorname{Spec}(A)$ and $U \hookrightarrow \operatorname{Spec}(B)$. Then there are $a \in A$ and $b \in B$ such that $A_a \cong B_b$.*

Proof. We may assume U is affine. It suffices to find $a \in A$ and $b \in B$ with $D(a), D(b) \subset U$ and $D(a) = D(b)$ as topological spaces. In that case the affine schemes $D(a)$ and $D(b)$ are isomorphic (all maps are open immersions), which yields the claim.

Write $U = \operatorname{Spec}(C)$. Let the maps induced by the open immersions be $A \xrightarrow{\phi} C$ and $B \xrightarrow{\psi} C$. Pick $a \in A$ with $D(a) \subset U$. We claim $A_a \cong C_{\phi(a)}$. Indeed, since $D(a) \subset U$, the map $A \to A_a$ factors as $A \to C \to A_a$, so by the universal property of localization we obtain inverse maps $A_a \leftrightarrows C_{\phi(a)}$.

Similarly, choose $b \in B$ with $D(b) \subset U$; then $B_b \cong C_{\psi(b)}$. Let $c = \phi(a)\psi(b) \in C$. We can thus find $\tilde{a} \in A$ and $\tilde{b} \in B$ whose images in $C_{\phi(a)}$ and $C_{\psi(b)}$ map to c up to a unit. Localizing then yields $A_{\tilde{a}} \cong C_c \cong B_{\tilde{b}}$. □

Proof of the previous proposition. We prove (1). Split the statement into two parts:

1. Step 1: For every affine open $U = \mathrm{Spec}(A) \subset Y$, the preimage $f^{-1}(U)$ is covered by affine opens $\mathrm{Spec}(B_k)$ with each B_k a finitely generated A-algebra. This is known for some affine open cover $U_i = \mathrm{Spec}(A_i)$ of Y. Set $V_i = U \cap U_i$, which covers U; note V_i need not be affine, and since U is quasi-compact we may assume only finitely many indices i occur. By the lemma, there exist finitely many $a' \in A_i$ and $a \in A$ with $A_{i,a'} \cong A_a$ such that the basic opens $\mathrm{Spec}(A_a) = \mathrm{Spec}(A_{i,a'})$ cover V_i. By assumption, $f^{-1}(\mathrm{Spec}(A_i))$ is covered by $\mathrm{Spec}(B_{ij})$, so $f^{-1}(\mathrm{Spec}(A_{i,a'})) = f^{-1}(\mathrm{Spec}(A_a))$ is covered by the $\mathrm{Spec}(B_{ij,a'})$. Each B_{ij} is a finitely generated A_i-algebra, hence $B_{ij,a'}$ is a finitely generated $A_{i,a'} \cong A_a$-algebra. Since A_a is a finitely generated A-algebra, each $B_{ij,a'}$ is a finitely generated A-algebra. All in all, this shows that $f^{-1}(U)$ is covered by opens of the form $\mathrm{Spec}(B_{ij,a'})$ where only finitely many i and $a' \in A_i$ occur and such that each $B_{ij,a'}$ is a finitely generated A-algebra.

2. Step 2: for any affine $\mathrm{Spec}(B) \subset f^{-1}(\mathrm{Spec}(A))$ we have that B is a finitely generated A-algebra. From (1) and the lemma, choose $b_1, \dots, b_n \in B$ with $(b_1, \dots, b_n) = 1$ and such that each B_{b_i} is a finitely generated A-algebra. Pick a relation $1 = \sum c_i b_i$ in B and a finite set $\mathscr{C} \subset B$ that generates each B_{b_i} as an A-algebra and contains $\{c_i\}$ and $\{b_i\}$. Let $B' \subset B$ be the A-subalgebra generated by $\mathscr{G}$. Then $c_i, b_i \in B'$, so the b_i generate the unit ideal of B' too. Viewing B as a B'-module, we have $B'_{b_i} \cong B_{b_i}$ since $\mathscr{C}$ generates each B_{b_i}; since the $D(b_i)$ cover $\mathrm{Spec}(B')$, it follows that $B' = B$. Hence B is finitely generated by $\mathscr{C}$ over A.

The same argument works for finite morphisms; we omit the details. ☐

Definition 5.13. A property $\mathscr{P}$ of morphisms of schemes is *stable under base change* if whenever $f : X \to Y$ has $\mathscr{P}$ and $Y' \to Y$ is any morphism, the base change $X_{Y'} \to Y'$ also has $\mathscr{P}$.

Clearly, being of finite type or finite is stable under base change. Let $f : X \to Y$ be of finite type. Then for every $y \in Y$, the fiber X_y is a scheme over $\kappa(y)$ covered by finitely many finitely generated $\kappa(y)$-algebras, so X_y is “almost a variety'' over $\kappa(y)$. The slogan is: morphisms of finite type = families of varieties.

Example 5.14. 1. Let $Q(x,y) = y^2 - x^3 - 7{\cdot}5\,x^2 + 7{\cdot}11$ and $R = \mathbb{Z}[x,y]/(Q(x,y))$. The generic fiber is $R_{\mathbb{Q}} = \mathbb{Q}[x,y]/(Q(x,y))$, an affine elliptic curve. For a prime p with residue field $\mathbb{F}_p$, the fiber over p is $\mathbb{F}_p[x,y]/(Q_p(x,y))$, where Q_p is the reduction of Q modulo p. For $p = 7$ we get $Q_p(x,y) = y^2 - x^3$, the cusp (additive reduction). For $p = 11$ we get $Q_{11}(x,y) =$

$y^2 - x^3 - 2x^2$, the node (multiplicative reduction). For all other primes we obtain an affine elliptic curve.

2. Let us look again at the example from the previous lecture. Let K be a field and $P(x, y, t) \in K[x, y, t]$. The map $K[t] \to R = K[x, y, t]/(P)$ is of finite type. For $t_0 \in K$, the fiber over $t - t_0$ is

$$\operatorname{Spec}\big(K[x, y]/(P(x, y, t_0))\big),$$

so set-theoretically we get a family $t_0 \mapsto V(P(x, y, t_0))$ of curves. Unwanted behaviors can occur: if some t_0 satisfies $P(x, y, t_0) = 0$, the fiber is the whole plane $\mathbb{A}^2_K$ (dimension jump). Degrees can also drop, e.g. for $P(x, y, t) = tx^3 + y + 1$ the fiber at $t = 0$ is a line, while generically it is a cubic.

This shows that maps of schemes do not always yields continuous families of varieties parametrized by the points of the base. To avoid such jumps and obtain continuity, one assumes flatness (an insight of Serre).

Definition 5.15. A ring map $A \to B$ is *flat* if B is a flat A-module, equivalently if $B_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}^c}$-module for every prime $\mathfrak{p}$ of B. A morphism of schemes is flat if for every Zariski open of the base, its preimage can be covered by affine opens that are flat over it. Equivalently, if for every $x \in X$ the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

We will study flatness in more depth later. For now, here is a key application.

5.4 Finite flat morphisms have fibers of the same cardinality

Let Y be an integral scheme and $f : X \to Y$ finite and flat. Let $x \in X$ with $y = f(x)$. Choose an affine open $U = \operatorname{Spec}(A) \subset Y$ containing y. Then $f^{-1}(U) = \operatorname{Spec}(B)$ is affine, $A \to B$ is integral, and B is a finitely generated A-module. If y corresponds to $\mathfrak{p} \subset A$, the fiber over y is $\operatorname{Spec}\big(B \otimes_A \kappa(\mathfrak{p})\big)$, a finite-dimensional $\kappa(\mathfrak{p})$-vector space, and we set

$$|X_{f(x)}| = \dim_{\kappa(f(x))}\big(B \otimes_A \kappa(f(x))\big).$$

Theorem 5.16. *If $f : X \to Y$ is finite and flat with Y integral, then $|X_{f(x)}|$ is independent of $x \in X$.*

Proof. Let $x \in X$, $y = f(x)$, and choose $U = \operatorname{Spec}(A) \subset Y$ with $y \in U$, writing $f^{-1}(U) = \operatorname{Spec}(B)$. Then x corresponds to a prime $\mathfrak{p} \subset B$ and y to $\mathfrak{p}^c \subset A$. Flatness implies $A_{\mathfrak{p}^c} \to B_{\mathfrak{p}^c}$ is flat.

Let now $(R, \mathfrak{m})$ be a local domain and let M be a finitely generated flat R-module. Note that M is torsion free (since R is a domain). Let k be the residue field of R and F be its fraction field. Then $M_k = M \otimes k$ is a finite dimensional k-vector space; fix a basis $\tilde{x}_1 \dots \tilde{x}_r$ of M_k and lift it to elements $x_i \in M$. By Nakayama's lemma then $x_1, \dots, x_r$ generate M as well. We need to show that they also form a basis of $M_F = M \otimes F$. But they generate M_F as an F-vector space, so we get $\dim_F(M_F) \leq \dim_k(M_k)$ and we need to prove equality. If $x_1, \dots, x_r$ were not linearly independent in M_F then there are some $a_i \in F$ not all zero such that $\sum r_i x_i = 0$. By clearing denominators we can assume that $r_i \in R$ and $\sum r_i x_i = 0$ in M. But then necessarily $r_i \in \mathfrak{m}$ for every i, for otherwise we would get a non-trivial relation in M_k. Pick the minimal n such that all $r_i \in \mathfrak{m}^n$ but there is some i with $r_i \notin \mathfrak{m}^{n+1}$. We can assume $i = 1$. So r_1 is non-zero in $\mathfrak{m}^n/\mathfrak{m}^{n+1}$, which is a finitely dimensional k-vector space. Now, by flatness the natural map $\mathfrak{m}^n \otimes M \to M \otimes R = M$ is injective. Consider the element $\sum_i r_i \otimes x_i \in \mathfrak{m}^n \otimes M$, which is sent to zero in M. Then it must be zero already in the tensor product. Pick any A-linear map $f : \mathfrak{m}^n/\mathfrak{m}^{n+1} \to k$ such that $f(r_1) \neq 0$ (note that A-linear in this context is equivalent to k-linear). So we get a map

$$\mathfrak{m}^n \otimes M \to (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes M \xrightarrow{f \otimes \mathrm{Id}_M} k \otimes M = M_k$$

which sends $0 = \sum_i r_i \otimes x_i \mapsto \sum_i f(r_i)\tilde{x}_i$. But then $f(r_i) = 0$ for every i necessarily, which is a contradiction. This shows that $\dim_F(M_F) = \dim_k(M_k)$, as required.

Now we conclude the proof. Let $A \subset B$ be an integral extension of rings; since in our case A is a domain, then also B is a domain. Pick a prime $\mathfrak{p} \subset B$ and consider the induced integral extension $A_{\mathfrak{p}^c} \subset B_{\mathfrak{p}^c}$. By assumption $A_{\mathfrak{p}^c}$ is a local domain and $B_{\mathfrak{p}^c}$ is a finitely generated flat $A_{\mathfrak{p}^c}$-module. So we use the previous computation to get the equality

$$\dim_{\kappa(\mathfrak{p}^c)} \kappa(\mathfrak{p}^c) \otimes_{A_{\mathfrak{p}^c}} B_{\mathfrak{p}^c} = \dim_{\mathrm{Frac(A)}} B \otimes_A \mathrm{Frac(A)}$$

but clearly $\kappa(\mathfrak{p}^c) \otimes_{A_{\mathfrak{p}^c}} B_{\mathfrak{p}^c} = \kappa(\mathfrak{p}^c) \otimes_A B$ which proves the result. ☐

This illustrates again why nilpotents matter. Consider K a field and $n \geq 1$. The map $\mathbb{A}^1_K \to \mathbb{A}^1_K$ induced by $K[t] \xrightarrow{t \mapsto t^n} K[t]$ is finite and free (hence flat). The fiber over $(t - t_0)$ is $K[t]/(t^n - t_0)$. If $\operatorname{char} K = p > 0$ and $n = p$, then for perfect K we can choose t_0' with $(t_0')^p = t_0$, and $K[t]/(t^p - t_0) \cong K[t]/(t - t_0')^p$ is nonreduced with a single point, but $\dim_K K[t]/(t - t_0')^p = p$ accounts for the "missing" points; if K is not perfect and t_0 is not a p-th power, then $K[t]/(t^p - t_0)$ is reduced but not geometrically reduced; if $(n, p) = 1$, then each fiber over

$t_0 \neq 0$ has n distinct points, while the fiber over $t_0 = 0$ is the single point 0 with ring $K[t]/(t^n)$ recording the lost points as before. This shows the concept of multiplicity.

Construction Let A be an integral domain with fraction field K, and let $K \subset L$ be finite. The integral closure B of A in L is finite over A, and $A \to B$ is integral. By uniqueness of integral closure, this sheafifies for integral schemes: if X is integral with function field $K(X)$ and $L/K(X)$ is finite, there exists a unique integral scheme $Y \xrightarrow{f} X$ with $K(Y) = L$ such that

- f is finite and the inclusion $K(X) \subset L$ identifies with the natural map $K(X) \subset K(Y)$ induced by the dominant morphism f.
- For every open affine $\operatorname{Spec}(A) \subset X$ the preimage $f^{-1}(\operatorname{Spec}(A)) = \operatorname{Spec}(B)$ is such that B is the integral closure of A in L.

Examples (Glimpses of ramification). As before, let A be a domain, $K = \operatorname{Frac}(A)$, and L/K a finite extension. Let B be the integral closure of A in L, so $A \subset B$ is integral. In general, $A \subset B$ need not be flat. For example, let $A = K[x,y]/(x^2 - y^3)$ be the affine cusp and $L = \operatorname{Frac}(A)$. Then the integral closure is $K[t]$ with $t^2 = y$ and $t^3 = x$ (indeed $t = x/y$), so $A \subset K[t]$ is integral. Let us show that it is not flat. We have $A_x \to K[t]_x = K[t]_t$ an isomorphism, so each fiber of $f : \operatorname{Spec}(A) \to \mathbb{A}^1_K$ has cardinality 1 over $D(t)$. To see nonflatness, consider $(t)^c = (x,y)$ and $A_{(x,y)} \to K[t]_{(t)}$. The maximal ideal of $A_{(x,y)}$ is not principal, so $\dim_K (x,y)/(x,y)^2 \geq 2$, whereas the maximal ideal of $K[t]_{(t)}$ is generated by t, so $\dim_K (t)/(t)^2 = 1$. Thus the map sends (x,y) into $(t)^2$, i.e. the morphism is *ramified* at (x,y) (this is in fact the definition of unramified morphism together with separability of residue extensions). One computes

$$\dim_K\big(A/(x,y) \otimes_A K[t]\big) = \dim_K K[t]/(t^2,t^3) = \dim_K K[t]/(t^2) = 2,$$

whereas over the function field the dimension is 1, so here ramification kills flatness.

Ramification need not obstruct flatness in general. Let $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ be the induced map (by going up, every prime of A is the contraction of some prime of B). If A is noetherian, $B_{\mathfrak{p}}$ has finitely many minimal primes $\mathfrak{q}_1, \dots, \mathfrak{q}_k$. Some $A_{\mathfrak{p}} \to B_{\mathfrak{q}_i}$ may be ramified, yet $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ can still be flat. For instance, if A is a Dedekind domain, then B is automatically flat: B is torsion-free and finitely generated over a Dedekind domain, whose local rings are PIDs, hence B is locally free and therefore flat (by the classification of finitely generated modules over PID;s). The miracle flatness theorem generalizes this to higher-dimensional regular rings.

For example, $K[t]$ is a Dedekind domain. Let $K(t) \subset L$ be a finite extension generated by $\alpha \in L$ with minimal polynomial $P(x) = x^n + a_{n-1}(t)x^{n-1} + \cdots + a_0(t) \in K(t)[x]$. Let B be as above. Choose a monic $b \in K[t]$ minimal (for divisibility) such that $bP(x) \in K[t,x]$. For a maximal ideal $\mathfrak{m} = (t - t_0)$, either $b(t_0) \neq 0$ or $b(t_0) = 0$. In the first case, α satisfies a monic equation over $K[t]_{\mathfrak{m}}$ because b is invertible in this ring. Since integral closure commutes with localization we deduce that α (although not necessarily integral over A) belongs to the image of $B \otimes_A A_{\mathfrak{m}}$ in $B_{\mathfrak{m}}$. Hence $B_{\mathfrak{m}}$ is generated by $1, \alpha, \dots, \alpha^{n-1}$ where $n = \deg(P)$, showing that this is locally free and hence flat over such ideal. On the other hand, if $b(t_0) = 0$ then α is not integral over $K[t]_{\mathfrak{m}}$. In general (as in the case of the cusp) we cannot do anything about this, but for Dedekind domains, we can choose another primitive element $\alpha' \in L$ whose corresponding b' satisfies $b'(t_0) \neq 0$. This is because the maximal ideal of $K[t]_{\mathfrak{m}}$ is generated by one element $\pi = t - t_0$; so if $b \in \mathfrak{m}$ then we can write $b = \pi^k b_0$ with $b_0 \notin \mathfrak{m}$ and by changing α to $\alpha' = \alpha/\pi^k$ the minimal polynomial becomes $x^n + \pi^k a_{n-1}(t) + \cdots + \pi^{nk} a_0(t)$ and so now b' has the required property. For the same reason, if the local rings of A are DVR's (i.e., Dedeking domains) then B is always automatically flat.

A geometric application of the primitive element theorem. Every finite field extension is generated by one element; this has a very precise geometric meaning. Let K be a field and let X/K be an algebraic variety (integral scheme of finite type over K; one often also assumes separable). We claim that X is birational to a hypersurface: there exists an affine open $U \subset X$ and a hypersurface $V(f) \subset \mathbb{A}_K^{d+1}$ with $d = \dim X$, together with an affine open $V \subset V(f)$, such that $U \cong V$. Write $U = \operatorname{Spec}(B)$ and apply Noether normalization to get a finite ring extension $K[x_1, \dots, x_d] \subset B$. Let F be the function field of B, so $K(x_1, \dots, x_d) \subset F$ is finite. Choose $\alpha \in F$ generating F with minimal polynomial $P(x) \in K(x_1, \dots, x_d)[x]$. After clearing denominators, assume $P(x) \in K[x_1, \dots, x_d][x]$, and consider the hypersurface $V(P(x)) \subset \mathbb{A}^{d+1}$. It may be reducible, but it has a unique irreducible component of dimension d (Gauss's lemma). Let the leading coefficient of P be $f(x_1, \dots, x_d)x^n$. Then P is monic in $K[x_1, \dots, x_d]_{\mathfrak{p}}[x]$ precisely when $\mathfrak{p} \in D(f) \subset \mathbb{A}_K^d$. The projection $\pi : V(P(x)) \to \mathbb{A}_K^d$ forgetting the last coordinate restricts by construction to a finite map $\pi^{-1}(D(f)) \to D(f)$ and the open $\pi^{-1}(D(f))$ is irreducible and lies in the unique d-dimensional component $V(f)$ of $V(P)$. But the function field of $\pi^{-1}(D(f))$, which is the function field of the hypersurface $V(f)$, is isomorphic to F by construction, and so by the bonus exercise, X and $V(f)$ are birational.

6 Lecture IX and X: More on flatness and generic properties

The letimotiv of these lectures is that ’morphisms behave nicely generically’. The next results will make it clear what I mean with this.

6.1 Generic flatness for finite and dominant morphisms

We show here that every finite dominant morphism $f : X \to Y$ of integral schemes is flat over an open subset of Y. Recall that dominant means that $f(X)$ is dense in Y.

Lemma 6.1. *Let $f : A \to B$ be a morphism of reduced rings. Then the induced morphism $\phi : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ has dense image if and only if f is injective.*

Proof. This is an easy exercise. Note that ϕ has dense image if and only if for every ideal $I \subset A$ such that the map ϕ factorises through $\operatorname{Spec}(A/I)$ we have that $I = 0$ since A is reduced. Now, if $A \to B$ is not injective, the image is contained in $V(\ker(f))$, so ϕ is not dominant. On the other hand, assume now that f is injective, and that $\operatorname{Im}(\phi) \subset V(I)$. But then this means that $I \subset \mathfrak{p}$ for every prime $\mathfrak{p} \subset B$; hence, that I consists of nilpotents. Impossible unless $I = 0$. □

Remark 6.2. As you will see in the exercises, if f is dominant (plus some other finiteness conditions, e.g., X and Y are noetherian or f is finitely presented) that $f(X)$ always contains an open subset of Y. This follows by Chevalley’s theorem which says that $f(X)$ is constructible.

Let now $f : X \to Y$ be a finite morphism of integral noetherian schemes and assume that it is dominant. We showed last lecture that it is not true in general that f is flat. We show now that f is flat over some open subset of Y. The result in fact follows by generic freeness for finitely generated modules:

Proposition 6.3. *Let R be a noetherian domain and let M be a finitely generated R-module. Then, there is some $f \in R$ such that M_f is free R_f-module.*

Proof. Let K be the fraction field of R and let M_K be the tensor product $M \otimes_R K$. Then M_K is a finite dimensional K-vector space. Let $m_1, \dots, m_r \in M$ be such that they form a basis in M_K. Let $\tilde{M} \subset M$ be the submodule generated by the various m_i. We show that there is some $f_1 \in R$ such that $\tilde{M}_{f_1} = M_{f_1}$. Let C be the cokernel of the inclusion. We know that $C_K = 0$. Moreover, we know that C is finitely generated. Fix generators $c_1, \dots, c_n \in C$. Then the fact that $C_K = 0$ implies that there is some $f_1 \in R$ such that $f_1 c_i = 0$ in C

for every i>. But then from the ses $0 \to \tilde{M} \to M \to C \to 0$ we get the ses $0 \to \tilde{M}_{f_1} \to M_{f_1} \to C_{f_1} = 0 \to 0$ proving the claim.

Next, we show that there is some $f_2 \in R_{f_1}$ such that $\tilde{M}_{f_1 f_2}$ is a free $R_{f_1 f_2}$ module. So we put $f = f_1 f_2$ and we prove the result.

By construction we have a surjection $R^r_{f_1} \twoheadrightarrow \tilde{M}_{f_1}$ which becomes an isomorphism once we tensor with K. We use the same trick as before with C being the kernel of the map above to conclude. We use that R is noetherian to say that this kernel if finitely generated. ☐

Note that it may be that $M_f = 0$. On the other hand, if $M_f \neq 0$ as soon as $M_K \neq 0$.

Corollary 6.4 (Slogan: every finite morphism is generically free/flat). *Let $f : X \to Y$ be a finite dominant map of integral noetherian schemes. Then there is an open subset $V \subset Y$ such that if $U = f^{-1}(V)$ then the induced map $f : U \to V$ is finite and flat (and surjective).*

Proof. We can assume that both X and Y are affine. So let $A \subset B$ be a finite extension of rings (integral + finitely generated) with B a domain. But then B is a finitely generated A-module and we use the previvous result to conclude and up to further localizing the induced map $A_a \to B_a$ is finite and B_a is free over A_a. Finally, since $f(X)$ is dense by assumption, using Chevalley we can also localize further and assume that $\mathrm{Spec}(A_a) \subset f(X)$. ☐

6.2 Generic freeness for finite type morphisms

We now wish to prove the same result for finite type morphisms. This is harder because in general the result from the previous sections are false for infinitely generated modules. So we really need to use that the module is an algebra in this case.

Theorem 6.5 (Generic flatness/finiteness for finite-type morphisms). *Let R be a noetherian domain and let $R \to S$ be a finite-type ring map with S also a domain. Let M be a finitely generated S-module. Then, there is a non-zero $f \in R$ such that M_f is a free R_f-module.*

Remark 6.6. The proof also works when S is not a domain.

The key to prove this, together with the results in the previous section, is the following variant of Noether normalization:

Proposition 6.7 (Generic Noether normalization). *Let $R \subset S$ be a finite-type ring extension. Assume that R is a domain. Then, there is a factorization*

$$R \to R[x_1, \dots, x_d] \subset S' \subset S$$

where S' is finite over $R[x_1, \dots, x_d]$ and $S'_f = S_f$ for some $f \in R \setminus 0$. In particular, there is an $f \in R \setminus 0$ such that $R_f[x_1, \dots, x_d] \subset S_f$ is finite.

Proof. Let $s_1, \dots, s_n$ be generators of S as an R-algebra. Let K be the fraction field of R and let S_K be the tensor product of S with K. Now, S_K is a finitely generated K-algebra, so by Noether-normalization we can find $K[x_1, \dots, x_d] \subset S_K$ finite. Up to multiplying the elements x_i by elements of K we can also assume that $x_i \in S$ for every i. Now, each s_i satisfies a minimal (monic) polynomial $P_i(T) \in K[x_1, \dots, x_d][T]$ and we can find some $f \in R$ such that $fP_i \in R[x_1, \dots, x_d][T]$ for every i. Since $fP_i(s_i) = 0$ in S_K we can also assume that f is such that $fP_i(s_i) = 0$ in S, by the usual property of the localization. Finally, let us put $s'_i = fs_i$ and let S' be the R-subalgebra of S generated by the various x_i and s'_i. We claim that this S' works. By construction each s'_i satisfies a monic equation over $R[x_1, \dots, x_d]$ (check this) hence S' is integral over $R[x_1, \dots, x_d]$. Moreover, $S_f = S'_f$ since S is generated by the various s_i as an R-algebra. This proves the result. □

Proof of Theorem 6.5. Let K be the fraction field of R and let S_K be as before. This is a finitely generated K-algebra so $\dim(S_K) \leq \infty$ by Noether normalization. So we shall prove the result by induction on $\dim(S_K)$. Now $S_K = 0$ if and only if $R \to S$ is not injective. In this case, any $f \in \ker(R \to S)$ would work. So we can assume that $R \to S$ is injective. By standard results in commutative algebra we can find a filtration $0 \subset M_1 \subset \dots M_n = M$ such that each quotient $M_i/M_{i-1} \cong S/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$ of S. Note that $S/\mathfrak{p} = S_K/\mathfrak{p}_K$ where $\mathfrak{p}_K = \mathfrak{p} \otimes_R K$ (this can be seen by tensoring the short exact sequence $0 \to \mathfrak{p} \to S \to S/\mathfrak{p} \to 0$ by K) and hence $\dim((S/\mathfrak{p})_K) = \dim(S_K/\mathfrak{p}_K) \leq \dim(S_K)$. Moreover, $S_K/\mathfrak{p}_K = S_K$ if and only if $\mathfrak{p}_K = 0$. But under our assumption that $R \to S$ is injective this can never happen because R is a domain. Also, by Krull principal ideal theorem (to be proven in the last lectures) we have that if $\mathfrak{p} \neq 0$ then $\dim(S_K/\mathfrak{p}_K) < \dim(S_K)$. Since an extension of free modules is free, we can thus assume that $S = M$.

Now, up to localizing R we can use the generic version of Noether normalization to assume that $R \subset R[x_1, \dots, x_d] \subset S$ with the last map being finite (we are replacing R with R_f for an appropriate $f \in R$). Let $s_1, \dots, s_k \in S$ be such that they form a basis of $\mathrm{Frac}(S)$ over $\mathrm{Frac}(R[x_1, \dots, x_d])$. We now get a short exact sequence

$$0 \to R[x_1, \dots, x_d]^k \xrightarrow{(s_1, \dots, s_k)} S \to N \to 0$$

of $R[x_1, \dots, x_d]$ modules, where injectivity comes from the fact that S is a domain. Also note that N is a finitely generated $R[x_1, \dots, x_d]$-module and that $N \otimes \mathrm{Frac}(R[x_1, \dots, x_d]) = 0$. But then by usual arguments there is some $g \in$

$R[x_1, \dots, x_d]$ such that $gN = 0$. So N is a module over $S' = R[x_1, \dots, x_d]/g$. Again by Krull principal ideal theorem we have that $\dim(S'_K) < \dim(S_K)$ and so by induction we can find some f such that N_f is a free R_f module. Using again that extensions of free modules is free, we conclude. □

There are many other similar theorems in algebraic geometry. These say that morphisms behave well over open subsets. In fact, this is a more general phenomenon which we will explain later.

6.3 Flat morphisms are open

A morphism is open if $f(U)$ is open for every $U \subset X$ open.

Recall from commutative algebra: A ring map $A \to B$ is flat if and only if for every primes $\mathfrak{p} \subset B$ the map $A_{\mathfrak{p}^c} \to B_{\mathfrak{p}}$ is flat if and only if for every prime $\mathfrak{q} \subset A$ the induced map $A_{\mathfrak{q}} \to B_{\mathfrak{q}}$ is flat. Also, recall that a flat map of local rings is faithfully flat and that flat maps satisfy going-down. Flat morphisms (plus some other finiteness conditions, the most general being finitely presented) are open maps. The standard way to prove this is to use Chevalley's theorem and the characterization of open/closed constructible sets. We shall use this approach later to study proper maps, so to not repeat ourselves we only write the key steps in proving the result:

Theorem 6.8. *Let $A \to B$ be a flat map of rings, and assume that A is noetherian and B is finitely-generated over A (or, more generally, only assume that B is a finitely presented A-algebra). Then the induced map $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ has open image. In particular, $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is an open mapping.*

Proof. The steps to prove this are as follows:

1. By Chevalley's theorem $f(\operatorname{Spec}(B))$ is constructible;
2. By going-down the set $f(\operatorname{Spec}(B))$ is also closed under generalizations, hence it is open;
3. Finally, for any $f \in B$ also B_f is finitely generated A-algebra, hence $f(\operatorname{Spec}(B_f))$ is open as well.

□

6.4 Remark on generic properties

Let $\mathscr{P}$ be a property on morphisms of schemes. One has a few definitions:

- $\mathscr{P}$ is local on the base if for any morphism $f: X \to Y$ if there is an open cover U_i of Y such that the induced morphism $f^{-1}(U_i) \to U_i$ has $\mathscr{P}$ for every i, then also f has $\mathscr{P}$. For example, flat or finite-type.
- $\mathscr{P}$ is stalk-local on the base if $f: X \to Y$ has $\mathscr{P}$ if and only if for every $y \in Y$ the induced map $X \times_Y \operatorname{Spec}(\mathscr{O}_{Y,y}) \to \operatorname{Spec}(\mathscr{O}_{Y,y})$ has $\mathscr{P}$ (e.g., smooth or flat morphisms);
- $\mathscr{P}$ is fibrewise on the base if $f: X \to Y$ has $\mathscr{P}$ if and only if for every $y \in Y$ the induced map $X_y = X \times_Y \operatorname{Spec}(\kappa(y)) \to \operatorname{Spec}(\kappa(y))$ has $\mathscr{P}$;
- $\mathscr{P}$ is local on the source if for any morphism $f: X \to Y$ if there is an open cover U_i of X such that the induced morphism $f_{|U_i} \to Y$ has $\mathscr{P}$ for every i, then also f has $\mathscr{P}$.
- $\mathscr{P}$ is 'fibrewise constructible' if for any morphism $f: X \to Y$ the set $\{y \in Y : X_y \to \operatorname{Spec}(\kappa(y)) \text{ has } \mathscr{P}\}$ is constructible;
- $\mathscr{P}$ is 'stalk constructible' if for any morphism $f: X \to Y$ the set $\{y \in Y : X \times_Y \operatorname{Spec}(\mathscr{O}_{Y,y}) \to \operatorname{Spec}(\mathscr{O}_{Y,y}) \text{ has } \mathscr{P}\}$ is constructible (e.g., open)

For example it turns out that most of the interesting properties (finite, flat, etale, smooth, etc) satisfy some of the above.

Proposition 6.9 (Application of the generic point). *Let $\mathscr{P}$ be a fibrewise (resp. stalk) constructible property of morphisms. Assume that $f: X \to Y$ is such that X and Y are notherian and Y is integral with generic point $\eta \in Y$. If $X_\eta \to \operatorname{Spec}(\kappa(\eta))$ has $\mathscr{P}$ then there is an open subset $U \subset Y$ such that $f^{-1}(U) \to U$ has $\mathscr{P}$.*

Proof. Note that since Y is reduced we have $\mathscr{O}_{Y,\eta} = \kappa(\eta)$. Now the set of points $y \in Y$ for which $\mathscr{P}(X_y \to y)$ holds (resp. for which $X \times_Y \operatorname{Spec}(\mathscr{O}_{Y,y}) \to \operatorname{Spec}(\mathscr{O}_{Y,y})$ holds) is a constructible subset of Y which contains the generic point. Hence it must contain an open subset $U \subset Y$. □

6.5 When are projective morphisms flat?

Let $A = \bigoplus_{d \geq 0} A_d$ be a graded domain generated by finitely many elements $x_1, \ldots, x_n \in A_1$. Recall that $\operatorname{Proj}(A) = \{\mathfrak{p} \subset A : \mathfrak{p} \text{ is homogeneous and } A_+ \not\subset \mathfrak{p}\}$ and that $\operatorname{Proj}(A) = \bigcup_{i=1}^n D_+(x_i)$ where $D_+(x_i) \cong \operatorname{Spec}(A_{x_i,0})$, the spectrum of the degree zero part of the localization A_{x_i}. The inclusion $A_0 \subset A$ induces an injection $A_0 \hookrightarrow \Gamma(\mathrm{Proj}(A), \mathcal{O}_{\mathrm{Proj}(A)})$ (which is in fact an isomorphism in many situations, see e.g. Hartshorne 'projective normality') and hence defines a morphism of schemes $\pi : \mathrm{Proj}(A) \to \mathrm{Spec}(A_0)$.

The fibre of π over a point $\mathfrak{p} \in \mathrm{Spec}(A_0)$ is then $\mathrm{Proj}(A \otimes_{A_0} A_0/\mathfrak{p})$ hence a projective scheme by definition. In fact, one can define projective morphisms of schemes as morphisms $f : X \to Y$ for which there is a Zariski open cover $\mathrm{Spec}(A_i) = U_i \subset Y$ such that the induced morphism $f^{-1}(\mathrm{Spec}(A_0)) \to \mathrm{Spec}(A_0)$ is of the form above (there are other definitions, more or less all equivalent). Projective morphisms are important because they describe families of projective varieties. Since we know by now that a family of algebraic varieties is well-behaved (read: continuous) when it is flat, the question we want to answer is: when is π flat? We answer this question when A_0 is a Dedekind domain (e.g. $A_0 = K[x]$.) Geometrically this translates into the family of projective varieties being parametrized by a one-dimensional (regular) scheme - i.e. a curve. For the general results one needs cohomology and other properties of Projective schemes, but the gist of it can already be seen in this particular case:

Theorem 6.10. *Assume that A_0 is a Dedekind domain. Then $\pi : \mathrm{Proj}(A) \to \mathrm{Spec}(A_0)$ is flat if and only if each A_d is a flat (= locally free) A_0-module.*

Proof. We know that $\mathrm{Proj}(A)$ is covered by $D_+(x_i)$ where $x_1, \dots, x_n \in A_1$ are generators of A. To show that π is flat is enough to show that $\pi_{|D_+(x_i)}$ is flat for every i (why?).

Now $D_+(x_i) = \mathrm{Spec}(A_{x_i,0})$ where $A_{x_i,0}$ is the degree-zero part of the localization of A at x_i. Let now $M_d = A_d/x_i^d \subset A_{x_i,0}$ (we use that A is a domain). So $\bigcup_d M_d = A_{x_i,0}$. Also, note that $M_d \cong A_d$ as a A_0-module via multiplication by x_i^d (again we use that A is a domain). Now we prove the result: assume that $A_{x_i,0}$ is flat and that A_d is not flat for some d. Then, there is an ideal $I \subset A_0$ such that $A_d \otimes I \to A_d$ is not injective. But since A_0 is a Dedekind domain every ideal is a flat module. So the maps $A_d \otimes I \to A_{d+1} \otimes I$ stay injective for every d. Hence also $A_{x_i,0} \otimes I \to A_{x_i,0}$ cannot be injective, since $(\varinjlim_d M_d) \otimes_{A_0} I = \varinjlim_d (M_d \otimes_{A_0} I)$ and each map in the colimit is injective (note that $M_d \cong A_d$). Hence also $A_{x_i,0} \otimes I \to A_{x_i,0}$ is not injective, which is a contradiction.

For the other direction, note that if A_d is flat for every d then also $\varinjlim_d M_d \cong \varinjlim_d A_d$ must be flat (as a colimit of flat modules is also flat). $\square$

Interpretation in terms of Hilbert function If now $A_0 = K$ is a field then $d \mapsto \dim_K(A_d)$ is called the Hilbert function of the graded ring A. The generating function associated to it is called the Hilbert series. It is well-known fact that $d \mapsto \dim_K(A_d)$ becomes a polynomial for $d >> 0$, which is called the Hilbert polynomial of A.

Corollary 6.11. *Let A be as in the previous theorem. Then if* $\mathrm{Proj}(A) \to \mathrm{Spec}(A_0)$ *is flat the Hilbert function of $A \otimes k(\mathfrak{p})$ does not depend on $\mathfrak{p} \in \mathrm{Spec}(A_0)$.*

In fact, the each A_d is flat and finitely generated A_0-modules - hence they are locally free. Since A_0 is integral this shows that the dimension of $A_d \otimes k(\mathfrak{p})$ as a $k(\mathfrak{p})$-vector space does not depend on the chosen prime.

Remark 6.12 (Constancy of rank and local freeness). If R is a Noetherian ring and M is a finitely generated R-module then M is locally free if and only if $\mathfrak{p} \mapsto \dim_{k(\mathfrak{p})}(M \otimes k(\mathfrak{p}))$ is constant.

The general form of the result above is:

Theorem 6.13. *The map $\pi : \mathrm{Proj}(A) \to \mathrm{Spec}(A_0)$ is flat if and only if the Hilbert polynomial of $A \otimes k(\mathfrak{p})$ does not depend on $\mathfrak{p} \in \mathrm{Spec}(A_0)$.*

Note that here we have the constancy of the Hilbert polynomial rather then of the Hilbert function. This means that not necessarily all the A_d's are flat. On the other hand, they become flat for $d >> 0$.

7 Lecture XI/XII

In the previous lectures and exercises we have shown that a flat morphism is universally open at least when f is of finite-presentation. Flat morphisms make a very general class of morphisms, and knowing that they are also (universally) open is very useful in practice. In this lecture we study the analogue condition: when is a morphism f of schemes closed? Let us recall some basic facts on schemes:

Lemma 7.1. *Let X be a scheme. Then, there is a natural bijection*

$$\{x \in X\} \longleftrightarrow \{Z \subset X : \ Z \text{ is closed and irreducible}\}$$

Proof. To any $x \in X$ we can associate its Zariski closure $Z = \bar{x}$. Conversely, to any irreducible and closed $Z \subset X$ we associate its generic point $\eta_Z \in X$. We recall how this is constructed: choose an open affine $U = \mathrm{Spec}(A)$ with $Z \cap U \neq \emptyset$. So $Z \cap U \subset U$ is an irreducible Zariski closed subset, and hence it corresponds uniquely to a prime ideal $I(Z) \subset A$. Then η_Z is the image of $I(Z) \in \mathrm{Spec}(A)$ under the natural inclusion $U \subset X$. It is easy to check that this is well-defined. □

Remark 7.2. Using the canonical reduced structure on closed subsets on a scheme X we can upgrade the correspondence above to

$$\{x \in X\} \longleftrightarrow \{Z \subset X : \ Z \text{ is closed integral subscheme}\}.$$

One thus have a natural order on X by saying that $x_1 \leq x_2$ if $x_1 \in \bar{x}_2$. One also says that x_2 specializes to x_1 or that x_1 generalizes to x_2.

Lemma 7.3. *Let $x \in X$. Then the image of* $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ *consists precisely of all points of X which specialize to x (whereas all the points of X that are specializations of x are clearly given by $\bar{x}$.*

Proof. Clearly, for any open affine $U = \mathrm{Spec}(A)$ with $x \in U$ we have $\mathcal{O}_{X,x} = \mathcal{O}_{U,x}$ and so we can assume that X is affine. Now if x corresponds to $\mathfrak{p} \subset A$ we have that $\mathrm{Spec}(A_{\mathfrak{p}}) = \{\mathfrak{q} \subset \mathfrak{p} \subset A\}$ which clearly proves the claim. ☐

Proposition 7.4. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then $f(X)$ is closed if and only if it is closed under specializations.*

Proof. The proof goes in three steps. Note also that this is more general than Chevalley's theorem as our only requirement is that f is quasi-compact (rather than of finite presentation).

- Step 1: To show that $f(X)$ is closed it is clearly enough to prove that for some affine open cover $\bigcup_i U_i = S$ we have that $f(X) \cap U_i$ is closed for every i. Since the induced maps $f^{-1}(U_i) \to U_i$ are also quasi-compact, we can reduce to when $S = \mathrm{Spec}(A)$ is affine.

- Step 2: We now prove the result when both S and $X = \mathrm{Spec}(B)$ are affine. In this case f is automatically quasi-compact. Now the image of f consists of all the contracted primes $\mathfrak{p}^c$ with $\mathfrak{p} \subset B$. Let $\mathfrak{q} \subset A$ be a prime with is in the closure of $f(X)$. This means that for every $f \notin \mathfrak{q}$ the principal open $D(f) = \mathrm{Spec}(A_f)$ intersects $f(X)$.

  But $f(X) \cap \mathrm{Spec}(A_f) = f(\mathrm{Spec}(B_f))$ which proves that $B_f \neq 0$ for every $f \ni n\mathfrak{q}$. Note that a localization is the zero ring if and only if f is nilpotent in B. But then also $B_{\mathfrak{q}} = B \otimes_A A_{\mathfrak{q}}$ is not the zero ring (for otherwise there would be some $f \in A \setminus \mathfrak{q}$ such that $B_f = 0$.)

  Finally, $f(X) \cap \mathrm{Spec}(A_{\mathfrak{q}}) = f(\mathrm{Spec}(B_{\mathfrak{q}})) \neq \emptyset$. But this means that there is some $\mathfrak{p} \subset \mathfrak{q} \subset B$ such that $\mathfrak{p}^c \subset \mathfrak{q}$. But since $f(X)$ is closed by specialization this implies that also $\mathfrak{q} \in f(X)$ - which proves the result.

- Step 3: now we prove the general statement when $S = \mathrm{Spec}(A)$ is affine. But then X must be covered by finitely many $U_i = \mathrm{Spec}(B_i)$ for $i = 1, \dots, n$.

Put $B = \prod B_i$ which is a ring (note: if i ranged over an infinite set, then B would still be a ring but of a very weird kind. Even assuming all the B_i's are fields it is hard to understand the resulting ring. For example it is not true that $\mathrm{Spec}(B) = \sqcup_i \mathrm{Spec}(B_i)$ when $n = \infty$). Now, B is a ring and $\mathrm{Spec}(B) = \bigsqcup_{i=1}^n U_i$ and so we have a factorization of the map $\mathrm{Spec}(B) \to X \to S$ where the first arrow is clearly surjective. So we use the previous point to conclude.

□

Next, one would like to understand when $f : X \to S$ is closed. This means that $f(Z)$ is closed for every closed subset $Z \subset X$.

Definition 7.5. Let $f : X \to S$ be a morphism of schemes. One says that specializations lift along f if for every $x \in X$ and $s \in S$ such that $s \leq f(x)$ then there is some $x' \in X$ such that $f(x') = s$ and $x' \leq x$.

Now one easily proves

Proposition 7.6. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is closed if and only if every specialization lifts along f.*

Proof. We prove one implication. Assume then that all specializations lift along f. We want to show that $f(Z)$ is closed for any closed subset $Z \subset X$. We can clearly assume that Z is irreducible, and we endow it with the unique reduced structure that makes it into a scheme. Now $Z \subset X$ is a closed immersion and hence it is quasi-compact. So also the restriction $f_{|Z} : Z \to S$ is quasi-compact. We only need to show that it is stable under specializations. So take $z \in Z$ and $s \in S$ be such that $s \leq f(z)$. We then can find some $x' \in X$ such that $f(x') = s$ and $x' \leq z$. But Z is clsoed – hence closed under specializations. So $x' \in Z$. This means that $f(Z)$ is closed under-specializations too, and hence closed. □

Specializations of points in a scheme X are closely related to morphisms from valuation rings to X.

7.1 Quick recap on valuation rings

Let A be a local domain with fraction field F. Let $B \subset F$ be a local ring.

Definition 7.7. We say that B dominates A (and we write $A \leq B$) if $A \subset B$ and $\mathfrak{m}_A = \mathfrak{m}_B \cap A$. We say that A is a valuation ring if A is maximal with respect to the domination relation.

These facts can be found in Atiyah-Macdonald:

1. Any local ring A is dominated in its fraction field by some valuation ring (use Zorn's lemma).
2. More interestingly: let A be a local domain with fraction field F and let $F \subset K$ be a field extension. Then there is a valuation domain $B \subset K$ with fraction field K and which dominates A.

Valuation rings are very weird objects, and we understand them well only when they are noetherian. In this case they are DVR, as we shall soon see.

Lemma 7.8. *Let A be a domain with fraction field F. Then A is a valuation ring if and only if for every $x \in F \setminus A$ we have $x^{-1} \in A$.*

Proof. Assume that A satisfies the condition above. If $A = K$ then A is a local ring by convention. If A is not K then it must contain a non-trivial maximal ideal $\mathfrak{m}$. We begin by showing that A is a local ring. Let $\mathfrak{m}_1, \mathfrak{m}_2$ be two maximal ideals of A and assume that $\mathfrak{m}_1 \neq \mathfrak{m}_2$. Then we can pick $x_1 \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ and $x_2 \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$. Consider the element $x = x_1/x_2$. Then if $x \in A$ we have that $x_1 \in (x_2)$ and in particular $x_1 \in \mathfrak{m}_2$, which shows that $\mathfrak{m}_1 = \mathfrak{m}_2$. If on the other hand $x \notin A$ then $x^{-1} \in A$ by the assumption, and we conclude as before. We now show that A is a valuation ring. Assume that $A \leq B$ with $B \subset K$. Pick $x \in \mathfrak{m}_B \setminus \mathfrak{m}_A$. Then $x \notin A$ for otherwise $x \in \mathfrak{m}_A$ since $\mathfrak{m}_A = \mathfrak{m}_B \cap A$. Hence $x^{-1} \in A$. But then $x^{-1} \in B$ and hence $1 \in \mathfrak{m}_B$ which is again a contradiction. □

Valuation rings take their name from the fact that we can attach to it a valuation in a natural way. Define the value group Γ of a valuation ring as $K^\times/A^\times$, with the group operation written additively. One then declares $\gamma_1 \geq \gamma_2$ if $\gamma_1 - \gamma_2$ is in the image of $A \setminus \{0\}$. One easily checks that Γ is a totally ordered abelian group. One then says that A id a DVR if $\Gamma \cong \mathbb{Z}$ with the natural ordering. For any $x \in K^\times$ one defines $v(x) \in \Gamma$ to be the induced element, which one calls the valuation of x. Sometimes one adds ∞ to Γ and declares $v(0) = \infty$. Note that $v(x) \geq 0$ if and only if $x \in A$. We state the following without proof.

Proposition 7.9. *1. A local domain A is a valuation ring if and only if every finitely generated ideal is principal;*

2. A valuation ring is a DVR if and only if it is noetherian.

7.2 Valuation rings and specializations

The use of valuation rings in algebraic geometry is given by the fact that they allow one to characterise specializations in terms of morphisms of schemes:

Proposition 7.10. *Let X be a scheme and $x_1 \leq x_2$ be points in X. Then there is a valuation ring A and a morphism $f :\ \mathrm{Spec}(A) \to X$ such that $f(\mathfrak{m}_A) = x_1$ and $f((0)) = x_2$.*

Proof. Let $Z = \bar{x}_2 \subset X$ be the integral closed subscheme associated to x_2. Then $x_1 \in Z$ hence we get an inclusion $\mathcal{O}_{Z,x_1} \subset k(x_2)$. Now, find any valuation ring $A \subset k(x_2)$ with $\mathcal{O}_{Z,x_1} \leq A$. Then the map of schemes given by the composition

$$\mathrm{Spec}(A) \to \mathrm{Spec}(\mathcal{O}_{Z,x_1}) \hookrightarrow Z \hookrightarrow X$$

clearly satisfies the requirements. ☐

Recall that last time we showed that for any specialization $x_0 \leq x_1$ there is a valuation ring V and a morphism $f :\ \mathrm{Spec}(V) \to X$ such that $f(0) = x_1$ and $f(\mathfrak{m}) = x_0$.

Remark 7.11. Valuation rings are pretty weird objects, and hard to understand in general. A valuation ring is noetherian if and only if $\Gamma = \mathbb{Z}$. In this case we have discrete valuation ring. Those are the local rings of regular varieties at codimension one points (divisors). For example since $K[x_1, \dots, x_n]$ is a UFD if f is an irreducible polynomial then the local ring $K[x_1, \dots, x_n]_{(f)}$ is a DVR.

If one adds some noetherianity hyphothesis one can state the valutative criterion for properness only with DVR.

The next result shows that the geometry of $\mathrm{Spec}(V)$ is only determined by the closed and the generic point of V:

Proposition 7.12. *To give a morphism $\mathrm{Spec}(V) \to X$ is equivalent to give two points $x_1 \leq x_2$ of X together with a map $K(x_2) \subset F$ such that $\mathcal{O}_{\bar{x}_2,x_1} \leq V$ via the inclusion $\mathcal{O}_{\bar{x}_2,x_1} \subset K(x_2)$.*

Proof. Let $f :\ \mathrm{Spec}(V) \to X$ be a morphism and let $x_1 = f(\mathfrak{m})$ and $x_2 = f(0)$. Then $x_1 \leq x_2$. Moreover the induced map $\mathrm{Spec}(F) \to X$ has image x_2 and so we get an inclusion $k(x_2) \subset F$. Let $\bar{x}_2$ be the closure of x_2. This is an integral closed subscheme and $x_1 \in \bar{x}_2$. Fact (exercise 3.11 d Hartshorne) the map $\mathrm{Spec}(V) \to X$ factorises via the closed immsersion $\bar{x}_2 \to X$. So now we obtain a diagram

$$\begin{tikzcd}
\mathcal{O}_{\bar{x}_2,x_1} \arrow[r, hook] \arrow[d, hook] & k(x_2) \arrow[d, hook] \\
V \arrow[r, hook] & F
\end{tikzcd}$$

and since this is induced by a morphism of locally ringed spaces, the maximal ideal $\mathfrak{m}$ of V intersects $\mathcal{O}_{\bar{x}_2,x_1}$ at $\mathfrak{m}_{x_1}$ and so V dominates it.

Let now $x_1 \leq x_2$ be as in the statement, so that we have a diagram

$$\begin{tikzcd}
\mathcal{O}_{\bar{x}_2,x_1} \arrow[r, hook] \arrow[d, hook] & k(x_2) \arrow[d, hook] \\
V \arrow[r, hook] & F
\end{tikzcd}$$

. Hence we get a map $\mathrm{Spec}(V) \to \mathrm{Spec}(\mathcal{O}_{\bar{x}_2,x_1})$. But then we get a map $\mathrm{Spec}(V) \to X$. By the commutativity of the diagram above it is easy to see that this sends 0 to x_2 and $\mathfrak{m}$ to x_1. □

7.3 Separated/proper morphisms

Recall now that in topology a space is T_2 if the diagonal $\Delta \subset X \times X$ is closed where the product space has the product topology. Now the point is that $X \times X$, as a scheme, has a bigger topology than just the product topology in general. So although if X is a scheme the diaganol in $X \times X$ with the product topology will never be closed, if we endow $X \times X$ with its Zariski topology, since we have more closed sets, we can hope that the definition works. And it does:

Definition 7.13. A morphism $f : X \to S$ is separated if the diagonal morphism $\Delta : X \to X \times_S X$ is a closed immersion.

In fact in this case being a closed immersion is a purely topological matter:

Lemma 7.14. *Δ is a closed immersion if and only if its image $\Delta(X) \subset X \times_S X$ is a closed subset.*

Proof. One implication is obvious. Assume now that $\Delta(X)$ is closed and let $\pi : X \times_S X \to X$ be one of the two projections. We have that the composition $X \xrightarrow{\Delta} \Delta(X) \xrightarrow{\pi} X$ is the identity, and since Δ is surjective, it must also be a homeomorphism onto its image, so $X = \Delta(X)$. So to show that Δ is a closed immersion it is enough to prove that for any $x \in X$ the map of sheaves $\mathcal{O}_{X \times_S X, \Delta(x)} \to \mathcal{O}_{X,x}$ is surjective. This is a purely local question, so we can prove the statement for any affine open of X: pick $\mathrm{Spec}(A) = U \subset X$ be an open and consider the diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & U \times_S U \arrow[d] \\
X \arrow[r] & X \times_S X
\end{tikzcd}$$

but now we know that Δ_U is locally the map induced by the mulitplication $A \otimes_B A \to A$ which is surjective (where $\mathrm{Spec}(B) \subset S$ and $\mathrm{Spec}(A) \subset f^{-1}(\mathrm{Spec}(B))$. □

The following is exercise 3.11 Hartshorne. I will not prove it.

Lemma 7.15. *Let $f : X \to Y$ be a morphidsm of schemes and assume that X is reduced. Let Z be a closed subscheme of Y such that $f(X) \subset Z$. Then f factorises through the closed immersion $Z \subset X$.*

So now we wrap it all up together:

Theorem 7.16. *Let $f : X \to Y$ be a quasi-compact morphism of schemes. For any valuation ring V with fraction field F*

- *f is universally closed if and only if for every valuation ring V with fraction field F and any diagram*

  $$\begin{tikzcd} \operatorname{Spec}(F) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(V) \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

  there exists a dotted arrow making it commute;

- *If X and S are notherian, then f is separated if and only if for any diagram as above, if the dotted arrow exists, it is unique;*

- *f is said to be proper if it is also of finite-type, separated, and universally closed.*

Proof.

- Assume that f is universally closed and consider the diagram. If we base-change X along the morphism $\operatorname{Spec}(V) \to S$ we get a new scheme X_V together with morphism $X_V \to \operatorname{Spec}(V)$. By assumption this is closed. Now, the morphisms $\operatorname{Spec}(F) \to X$ and $\operatorname{Spec}(F) \to \operatorname{Spec}(V)$ give a morphism to the fibre product by the universal property $\operatorname{Spec}(F) \to X_V$. Hence the image of $X_v \to \operatorname{Spec}(V)$ contains the generic point (0) and so it must be the whole thing since it is closed. So there is a point $x_1 \in X_V$ which maps to $\mathfrak{m} \in \operatorname{Spec}(V)$. Moreover, this point is the specialization of the image x_2 of $\operatorname{Spec}(F)$. So $x_1 \leq x_1$ and $f(x_1) = \mathfrak{m}$ and $f(x_2) = (0)$. So we get a diagram as before

  $$\begin{tikzcd} \mathcal{O}_{\bar{x}_2, x_1} \arrow[r, hook] \arrow[d, hook] & k(x_2) \arrow[d, hook] \\ V \arrow[r, hook] & F \end{tikzcd}$$

  which describes a morphism $\operatorname{Spec}(V) \to X_V$ hence a morphism $\operatorname{Spec}(V) \to X$. This is easily seen to satisfy the hyphothesis.

To prove the other way, let $S' \to S$ be any morphism and consider the base-change $X_{S'} \to S'$. We want to show that this is closed, i.e., that specialization lifts. Take any points $s_1 \leq s_2$ is S' such that $s_2 = f(x_2)$. Let $F = k(x_2)$. Then we have an inclusion $k(s_2) \subset F$. Now the local ring $\mathcal{O}_{\bar{s}_2,s_1} \subset k(s_2) \subset F$ can be dominated by a valuation with $V \subset F$ with fraction field F. So this gives a map $\operatorname{Spec}(V) \to S'$ mapping $(0) \mapsto s_2$ and $\mathfrak{m} \mapsto s_1$. This also gives a map $\operatorname{Spec}(F) \to X_{S'}$ which is nothing but the inclusion of x_2. So we get a diagram

$$\begin{tikzcd}
\operatorname{Spec}(F) \arrow[r] \arrow[d] & X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(V) \arrow[r] \arrow[urr, dotted, "\exists"] & S' \arrow[r] & S
\end{tikzcd}$$

and by assumption the arrow exists. But then by the universal property of the fibre product also an arrow

$$\begin{tikzcd}
\operatorname{Spec}(F) \arrow[r] \arrow[d] & X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(V) \arrow[r] \arrow[ur, dotted, "\exists"] & S' \arrow[r] & S
\end{tikzcd}$$

exists. The image of $\mathfrak{m}$ under this map is then the required lifting of s_1.

- Assume that $\Delta(X)$ is closed and that there are two arrows

$$\begin{tikzcd}
\operatorname{Spec}(F) \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(V) \arrow[r] \arrow[ur, bend left, "e_2"] \arrow[ur, bend right, "e_1"'] & S
\end{tikzcd}$$

making the diagram commute. Then we get a diagram

$$\begin{tikzcd}
\operatorname{Spec}(V) \arrow[r, "e_1"] \arrow[d, "e_2"] & X \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

hence a morphism $\operatorname{Spec}(V) \to X{\times}_S X$. But since e_1, e_2 resitrict to the same morphism on $\operatorname{Spec}(F)$ then the map above sends (0) in $\Delta(X)$. But the image of $\mathfrak{m}$ is a specialization of the image of (0) hence by closedness also $\mathfrak{m}$ is in $\Delta(X)$. But this means that $e_1 = e_2$.

On the contrary, assume that uniqueness if existence is satisfied and that $\Delta(X)$ is not closed. But $\Delta(X)$ is constructible hence there is a point $x_2 \in \Delta(X)$ and a point $x_1 \in X \times_S X$ such that $x_1 \leq x_2$ and $x_1 \notin \Delta(X)$. We know that there is a valaution ring and a map $h : \operatorname{Spec}(V) \to X \times_S X$ such that $(0) \mapsto x_2$ and $\mathfrak{m} \mapsto x_1$. But then we get two morphisms $h_1, h_2 : \operatorname{Spec}(V) \to X$. These two morphisms send the generic point to the same point since $h(0) \in \Delta(X)$ but they cannot be the same morphism for otherwise also $h(\mathfrak{m}) \in \Delta(X)$.

□

We already know that morphisms of affine schemes are separated.

Projective space is proper

Proposition 7.17. *The morphism* $\mathbb{P}^n_{\operatorname{Spec}(\mathbb{Z})} \to \operatorname{Spec}(\mathbb{Z})$ *is proper.*

The proof is very easy once we have the following

Lemma 7.18. *Let R be any domain. Then there is a natural injection*

$$\{(r_i)_{i=0,\dots,n} \in R^{n+1} : (r_0, \dots, r_n) = R\}/R^\times \to \operatorname{Mor}(\operatorname{Spec}(R), \mathbb{P}^n_{\mathbb{Z}}).$$

(in fact, this is an isomorphism if R has trivial class group). Moreover, this is an isomorphism if R is a field.

Proof. Let r_i be a tuple as in the statement. Let R_i be the localization of R at r_i. Write $S = \mathbb{Z}[x_0, \dots, x_n]$ and consider the maps $S_{x_i,0} \to R_i$ sending $x_j/x_i \mapsto r_j/r_i$. This gives morphisms $\operatorname{Spec}(R_i) \to D_+(x_i)$. Moreover, they glue on the intersections, and since $\bigcup_i \operatorname{Spec}(R_i) = \operatorname{Spec}(R)$ we conclude.

A comment on glueing. the ring $S_{x_i,0}$ is generated by elements $t_{j,i} = x_j/x_i$. In $S_{x_ix_j,0}$ the identification given by the glueing then says $t_{j,i}t_{i,j} = 1$. So the morphisms glue if and only if $r_i/r_jr_j/r_i = 1$ in R_{ij} which is obvious.

Assume now that R is a field F. Let $f : \operatorname{Spec}(F) \to \mathbb{P}^n$ with image x. Then there is some i such that $x \in D_+(x_i)$. So f is uniquely determined by a morphism $S_{x_i,0} \to F$ for some i. Any such morphism yields a tuple $(f_0, \dots, 1, \dots, f_n)$ and it is easy to show the statement from here. □

Proof. Consider a diagram

$$\begin{tikzcd}
\operatorname{Spec}(F) \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\
\operatorname{Spec}(V) \arrow[r] & \operatorname{Spec}(\mathbb{Z})
\end{tikzcd}$$

by the previous lemma the morphism on the top corresponds to a tuple $(f_0, \dots, f_n) \in K^{n+1} \setminus 0$. Pick i such that $v(f_i)$ is the smallest. Then $f_j/f_i \in V$ for every j. Hence we get a tuple $(r_0, \dots, 1, \dots, r_n)$ which gives a morphism $\mathrm{Spec}(V) \to \mathbb{P}^n_{\mathbb{Z}}$. Uniqueness is also easy from these descriptions, once you show that $\mathrm{Spec}(V)$ necessarily needs to land in an affine open chart. □

8 Lecture XIII/ XIV

The aim of these last two lectures is to introduce regular local rings and regular schemes. These are the schemes whose local structure we understand the best, and there are even algebraic ways to say that regular schemes 'look locally as $\mathbb{A}^n$' e.g. by using the etale topology or formal completions. We begin by proving the following fundamental result:

Theorem 8.1. *Let $(A, \mathfrak{m})$ be a noetherian local ring with residue field k. Then*

$$\dim(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2).$$

This will require some time and some work in commutative algebra.

8.1 Artin-Rees lemma

Let A be a ring $I \subset A$ an ideal and put $\tilde{A} := \bigoplus_{d \geq 0} I^d$ where $I^0 = A$ by convention. We make this into a (graded) ring in the natural way.

Remark 8.2. If I is finitely generated then also $\tilde{A}$ is a finitely generated A-algebra via the surjection $A[x_1, \dots, x_n] \to \tilde{A}$ sending $x_k \mapsto i_k \in I = \tilde{A}_1$ where $I = (i_1, \dots, i_n)$. In particular, if A is noetherian, then $\tilde{A}$ is automatically noetherian.

Definition 8.3. Let M be an A-module.

- An I-filtration on M is a decreasing filtration $\cdots M_{n+1} \subset M_n \subset \cdots M_1 \subset M_0 = M$ such that $IM_n \subset M_{n+1}$;
- The filtration is called stable if $M_{n+1} = IM_n$ for $n \gg 0$.

Stable filtrations are easy to characterise:

Lemma 8.4. *Let A be noetherian and M be finitely generated. Then $M_\bullet \subset M$ is stable if and only if $\tilde{M} = \bigoplus_{d \geq 0} M_d$ is a finitely-generated $\tilde{A}$-module.*

Proof. For $n \geq 0$ put $N_n := \bigoplus_{d \leq n} M_d$ and $P_n := N_n \oplus (\bigoplus_{d \geq 1} I^d M_n)$ and note that $I^d M_n \subset M_{n+d}$ so that $P_n \subset \tilde{M}$ is an increasing sequence of $\tilde{A}$-submodules. Now if $\tilde{M}$ is finitely generated, since $\tilde{A}$ is noetherian too, this sequence must stabilize and so $P_n = \tilde{M}$ for $n \gg 0$, which implies $IM_n = M_{n+1}$ for $n \gg 0$.

On the other hand, if $M_\bullet$ is stable, then $P_n = \tilde{M}$ for $n \gg 0$. Since P_n is finitely generated $\tilde{A}$-module for every n we conclude. ☐

Lemma 8.5. *Set-up as before, $M_\bullet \subset M$ a stable filtration, $N \subset M$ a submodule. Then $N_i := M_i \cap N$ is also a stable I-filtration on N.*

Proof. In fact, $\tilde{N} \subset \tilde{M}$ is a $\tilde{A}$-submodule by construction. Since $\tilde{A}$ is noetherian and $\tilde{M}$ is finitely generated also $\tilde{N}$ is finitely generated – hence the result. ☐

Corollary 8.6 (Artin-Rees lemma). *Let A be a noetherian ring, M a finitely generated A-module, $N \subset M$ a submodule. Then*

$$I^{n+1}M \cap N = I(I^n M \cap N)$$

for $n \gg 0$.

Proof. Simply take $M_n = I^n M$ in the previous result. ☐

Theorem 8.7 (Krull). *Let A be a noetherian ring, M a finitely generated A-module, $I \subset A$ an ideal. Then*

$$\bigcap_{d \geq 0} I^d M = \{m \in M : \exists i \in I : (1+i)m = 0\}.$$

Proof. We prove both inclusion. We begin with the easiest: take $m \in M$ such that $(1+i)m = 0$ for some $i \in I$. Then $m = -im$ hence $m \in IM$ hence by induction $m \in I^d M$ for every $d \geq 0$, which proves that m belongs to the intersection on the left hand side.

Take now $m \in \bigcap_{d \geq 0} I^d M$ and consider the cyclic submodule $N = Am \subset M$. Apply Artin-Rees lemma and obtain $I^{n+1}M \cap N = I(I^n M \cap N)$ for $n \gg 0$. Now by assumption $m \in I^d M \cap N$ for every d and, moreover, any element of $I^n M \cap N$ is of the form am for some $a \in A$ by construction. Hence by the equality above we get $m = i(am)$ for some $i \in I$ and $a \in A$ or $(1 - ia)m = 0$ which proves the result. ☐

Remark 8.8. In particular if M is torsion free then $\bigcap_{d \geq 0} I^d M = 0$. For instance, if A is a domain, then $\bigcap_{d \geq 0} I^d A = 0$ for every proper ideal $I \subset A$.

8.2 Krull principal ideal theorem

We recall the following characterization of Artinian rings:

Proposition 8.9. *Let $(A, \mathfrak{m})$ be a noetherian local ring. Then the following are equivalent:*

1. *A is artinian;*
2. $\dim(A) = 0$*;*
3. $\mathfrak{m}^k = 0$ *for some $k \geq 1$;*
4. $\mathfrak{m} = \sqrt{0}$.

We now prove

Theorem 8.10 (Krull Hauptidealsatz). *Let A be a noetherian ring, $f \in A$ not invertible. Let $\mathfrak{p} \subset A$ be minimal among the primes with $f \in \mathfrak{p}$ (i.e., a minimal prime of A/f). Then* $\mathrm{ht}(\mathfrak{p}) \leq 1$.

Recall that $\mathrm{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}})$.

Remark 8.11. In the situation before we have that $V(f) \subset \mathrm{Spec}(A)$ is a closed integral subscheme cut out by a single equation, i.e., an hypersurface. If A is catenary, then Krull's HIS says that the dimension of $V(f)$ is either the dimension of $\mathrm{Spec}(A)$ or one less.

Proof. We make some reductions:

- We may assume that A is local with maximal ideal $\mathfrak{p}$ since $\mathrm{ht}_A(\mathfrak{p}) = \mathrm{ht}_{A_{\mathfrak{p}}}(\mathfrak{p}A_{\mathfrak{p}})$;
- We may assume that A is also a domain since $\mathrm{ht}_A(\mathfrak{p}) = \mathrm{ht}_{A/\mathfrak{q}}(\mathfrak{p})$ where $\mathfrak{q}$ is any minimal prime appearing in a chain of maximal length computing the height.

So we need to prove: let $(A, \mathfrak{p})$ be a noetherian local domain and assume that there is $f \in \mathfrak{p}$ such that $f \notin \mathfrak{q}$ for any proper ideal $\mathfrak{q} \subsetneq \mathfrak{p}$. Then $\dim(A) \leq 1$ (recall the remark above).

So let us look at A/f. Now if $\mathfrak{q} \subset A/f$ is a prime ideal then this yields a prime $\mathfrak{q} \subset A$ with $f \in \mathfrak{q}$ and so necessarily $\mathfrak{q} = \mathfrak{p}$ by assumption. Hence $\dim(A/f) = 0$, so A/f is artinian. Let now $\mathfrak{q} \subsetneq \mathfrak{p}$ be any prime ideal. We need to show that $\mathfrak{q} = 0$. Define $\mathfrak{q}_n := \mathfrak{q}^n A_f \cap A$ – where the intersection is taken inside the fraction field of A. Then $\mathfrak{q}_n \subset A$ is an ideal which satisfies

$\mathfrak{q}_n A_f = \mathfrak{q}^n A_f$. Moreover $\mathfrak{q}^n \subset \mathfrak{q}_n$ clearly. Also, note that since $f \notin \mathfrak{q}$ by assumption the ideal $\mathfrak{q}_n$ is not the trivial ideal.

Now, the images of $\mathfrak{q}_n$ in A/f yield a decreasing sequence of ideals (namely, $(\mathfrak{q}_n + (f))/f$) hence, this sequence must stabilize for n big enough. That is, $(\mathfrak{q}_n + (f))/f = (\mathfrak{q}_{n+1} + (f))/f$ for $n \geq n_0$ hence $\mathfrak{q}_n + (f) = \mathfrak{q}_{n+1} + (f)$ for $n \geq n_0$. In particular $\mathfrak{q}_n \subset \mathfrak{q}_{n+1} + (f)$ for $n \geq n_0$. Let now $x \in \mathfrak{q}_n$, then there is $y \in A$ such that $x - fy \in \mathfrak{q}_{n+1}$ hence $fy \in \mathfrak{q}_n$ since $\mathfrak{q}_{n+1} \subset \mathfrak{q}_n$. We now claim that also $y \in \mathfrak{q}_n$. But since f is invertible in A_f we have that $y \in \mathfrak{q}^n A_f$ and also $y \in A$ so $y \in \mathfrak{q}_n = \mathfrak{q}^n A_f \cap A$. Hence, $\mathfrak{q}_n \subset \mathfrak{q}_{n+1} + f\mathfrak{q}_n \subset \mathfrak{q}_{n+1} + \mathfrak{p}\mathfrak{q}_n$ hence $\mathfrak{q}_n = \mathfrak{q}_{n+1}$ by Nakayama. Thus $\mathfrak{q}_n = \mathfrak{q}_{n+1}$ for every $n \geq n_0$ hence $\mathfrak{q}_n A_f = \mathfrak{q}^n A_f = \mathfrak{q}^{n_0} A_f$ for every $n \geq n_0$ which implies that $\mathfrak{q}^{n_0} A_f = \bigcap_{n \geq n_0} \mathfrak{q}^n A_f = 0$ where the last equality follows by Krull's theorem in the previous section (since A_f is a domain). Hence $\mathfrak{q} A_f = 0$ hence $\mathfrak{q} = 0$ as desired. □

We now aim at obtaining more precise results. We need the following lemma:

Lemma 8.12. *Let A be a noetherian ring, $f \in A$. Then for any chain of primes*

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

such that $f \in \mathfrak{p}_n$ there is another chain

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n$$

with $\mathfrak{q}_n = \mathfrak{p}_n$ and $f \in \mathfrak{q}_1$.

Proof. By induction on n. If $n = 1$ there is nothing to prove. So let $n > 1$. We can assume $f \notin \mathfrak{p}_{n-1}$ for otherwise we can just apply induction. Let now $\mathfrak{q}_{n-1}$ be any minimal prime among those satisfying $\mathfrak{p}_{n-2} + (f) \subset \mathfrak{q} \subset \mathfrak{p}_n$. Note that $\mathfrak{p}_{n-2} \subsetneq \mathfrak{q}_{n-1}$ since $f \notin \mathfrak{p}_{n-2}$ and $f \in \mathfrak{q}_{n-1}$. Now we apply induction to the chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_{n-2} \subsetneq \mathfrak{q}_{n-1}$ and we obtain a sequence $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_{n-1}$ with $f \in \mathfrak{q}_1$. Now, the image of $\mathfrak{q}_{n-1}$ in $A/\mathfrak{p}_{n-2}$ has height either zero or one by Krull's theorem. But this cannot have height zero since $(0) \subsetneq \mathfrak{q}_{n-1}/\mathfrak{p}_{n-2}$ hence it must have height one. On the other hand, the image of $\mathfrak{p}_n$ in $A/\mathfrak{p}_{n-2}$ has height ≥ 2 since we have the chain $(0) \subsetneq \mathfrak{p}_{n-1}/\mathfrak{p}_{n-2} \subset \mathfrak{p}_n/\mathfrak{p}_{n-2}$ by assumption. So $\mathfrak{q}_{n-1}$ cannot be $\mathfrak{p}_n$ hence $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_{n-2} \subsetneq \mathfrak{q}_{n-1} \subsetneq \mathfrak{p}_n$ is the required chain. □

Corollary 8.13 (Height/codimension estimate). *Let A be a noetherian ring and let I be an ideal generated by r elements. Let $\mathfrak{p}$ be minimal among the primes satisfying $I \subset \mathfrak{p}$. Then* $\mathrm{ht}(\mathfrak{p}) \leq r$.

Proof. By induction on r. If $r = 1$ this is Krull HIS. So suppose $r \geq 2$ and let $f_1, \dots, f_r$ be generators of I. Now $I/f_r \subset A/f_r$ is generated by $r - 1$ elements and so by induction $\mathrm{ht}(\mathfrak{p}/f_r) \leq r - 1$. Now let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$ be any chain of primes. Since $f_r \in \mathfrak{p}_n$ we can find a new chain of the same length $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n = \mathfrak{p}$ with $f_r \in \mathfrak{q}_1$. Hence we obtain a chain $\mathfrak{q}_1/f_r \subsetneq \cdots \subsetneq \mathfrak{q}_n/f_r = \mathfrak{p}/f_r$ of primes in A/f_r which shows that $n - 1 \leq \mathrm{ht}_{A/f_r}(\mathfrak{p}/f_r) \leq r - 1$ hence $n \leq r$ as wanted. ☐

In particular, we obtain as a corollary the result claimed at the beginnin of the lecture:

Corollary 8.14. *Assume that* $(A, \mathfrak{m})$ *is a noetherian local ring. Then*

$$\dim(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2).$$

Proof. By Nakayama $\mathfrak{m}$ is generated by $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ elements. Hence $\dim(A) = \mathrm{ht}(\mathfrak{m}) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. ☐

We conclude the result with a better estimate of the dimension of hypersurfaces (= subschemes cut out by a single equation)

Proposition 8.15. *Let* $(A, \mathfrak{m})$ *be a local noetherian ring,* $f \in \mathfrak{m}$. *Then* $\dim(A/f) \geq \dim(A) - 1$ *and equality holds when* f *is not contained in any minimal prime of* A.

Proof. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{m}$ be a chain of primes. Since $f \in \mathfrak{m}$ we can find a new chain $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n = \mathfrak{m}$ with $f \in \mathfrak{q}_1$ hence $\mathfrak{q}_1/f \subsetneq \cdots \subsetneq \mathfrak{q}_n/f = \mathfrak{m}/f$ is chain of A/f showing that $\dim(A/f) \leq n - 1$.

Now assume that f is not contained in any minimal prime of A, we want to show that the inequality above is an equality. But if $\mathfrak{p} \subset A$ is minimal among the primes satisfying $f \in \mathfrak{p}$ we must have by Krull that $\mathrm{ht}(\mathfrak{p}) = 0$ or 1. This cannot be zero by assumption, so $\mathrm{ht}(\mathfrak{p}) = 1$. Hence $\dim(A/f) \leq \dim(A) - 1$ which proves the result. ☐

Make an example when the hypothesis above are needed to grant equality.

8.3 Regular schemes

Definition 8.16. Let $(A, \mathfrak{m})$ be a noetherian local ring with residue field k. We say that A is regular if $\dim(A) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.

If X is a scheme, we say that X is regular at $x \in X$ if $\mathcal{O}_{X,x}$ is a regular local ring. We say that X is regular if it is regular at all its points.

For example, any DVR is a regular ring.

Interpretation of $\mathfrak{m}/\mathfrak{m}^2$ as tangent space, informal Let K be any field and let $x_0 \in \mathbb{A}^n_K(K) = K^n$ be a K-rational point (that is, a morphism $\operatorname{Spec}(K) \to \mathbb{A}^n_K$). Up to translation we can assume $x_0 = 0$ but this is not necessary. Let $\mathcal{O}_{\mathbb{A}^n_K, x_0}$ be the local ring of $\mathbb{A}^n_K$ at x_0. So in our case this is the localization of $K[x_1, \dots, x_n]$ at $(x_1, \dots, x_n)$. Now choose a vector direction $\vec{v} \in K^n$. To this we can associate the diretional derivative

$$D_{x_0,\vec{v}}(f) = \text{“}\lim_{\epsilon \to 0} \frac{f(x_0 + \epsilon\vec{v}) - f(x_0)}{\epsilon}\text{”} = \sum_i \frac{df}{dx_i}(x_0) v_i$$

for any $f \in \mathcal{O}_{\mathbb{A}^n_K, x_0}$, where $\vec{v} = (v_1, \dots, v_n)$. Note that no limit is really necessary since f is a rational function, so that the derivative can be taken using purely algebraic rules. Now the function $D_{x_0,\vec{v}}$ satisfies certain properties:

1. $D_{x_0,\vec{v}}$ is K-linear in $\vec{v}$;
2. $D_{x_0,\vec{v}}(f)$ is K-linear in f;
3. $D_{x_0,\vec{v}}(k) = 0$ for every $k \in K$;
4. (Leibniz rule) $D_{x_0,\vec{v}}(fg) = D_{x_0,\vec{v}}(f)g(x_0) + f(x_0)D_{x_0,\vec{v}}(g)$.

Since $D_{x_0,\vec{v}}(K) = 0$ for any $f \in \mathcal{O}_{\mathbb{A}^n_K, x_0}$ we also have $D_{x_0,\vec{v}}(f) = D_{x_0,\vec{v}}(f - f(x_0))$ and $f - f(x_0) \in \mathfrak{m}_{x_0} \subset \mathcal{O}_{\mathbb{A}^n_K, x_0}$. Thus $D_{x_0,\vec{v}}$ is completely determined by its behavior as a function on $\mathfrak{m}_{x_0}$.

Lemma 8.17. *The derivation $D_{x_0,\vec{v}} : \mathfrak{m}_{x_0} \to K$ sends $\mathfrak{m}^2_{x_0}$ to zero and hence induces a K-linear function $D_{x_0,\vec{v}} : \mathfrak{m}_{x_0}/\mathfrak{m}^2_{x_0} \to K$.*

Proof. The proof is very easy: since $\mathfrak{m}^2_{x_0}$ is generated by fg with $f, g \in \mathfrak{m}_{x_0}$ hence by the Leibniz rule

$$D_{x_0,\vec{v}}(fg) = D_{x_0,\vec{v}}(f)g(x_0) + f(x_0)D_{x_0,\vec{v}}(g) = 0 + 0$$

since $f(x_0) = g(x_0) = 0$. □

In fact, we also have

Proposition 8.18. *There is a natural one-to-one correspondence between $(\mathfrak{m}_{x_0}/\mathfrak{m}^2_{x_0})^\vee$ and functions $D : \mathcal{O}_{\mathbb{A}^n_K, x_0} \to K$ satisfying the properties (2), (3) and (4) above.*

Proof. The previous lemma tells us how to obtain an element of $(\mathfrak{m}_{x_0}/\mathfrak{m}^2_{x_0})^\vee$ from such a D. To go the other way, consider $\phi \in (\mathfrak{m}_{x_0}/\mathfrak{m}^2_{x_0})^\vee$ and define $D(f) = \phi(f - f(x_0)$. It is easy to show that this is satisfies (2), (3) and (4). □

In fact, any such D must be of the form $D_{x_0,\vec{v}}$ for some $\vec{v}$:

Proposition 8.19. *There is a natural one-to-one correspondence between K^n and functions $D : \mathcal{O}_{\mathbb{A}^n_K,x_0} \to K$ satisfying the properties (2), (3) and (4) above.*

Proof. Given $\vec{v} \in K^n$ we know how to construct D. Let us go the other way around and let D be given. Let $x_0 = (a_1, \dots, a_n)$ with $a_i \in K$ so $\mathfrak{m}_{x_0}$ is generated by the various linear functions $x_i - a_i$. Now note that $D_{x_0,\vec{v}}(x_i - a_i) = v_i$ hence we define $\vec{v} := (D(x_1 - a_1), \dots, D(x_n - a_n))$ and check that this yields the required bijection. ☐

So, if we interpret K^n as the tangent space of $\mathbb{A}^n_K$ at x_0 we have three natural identifications

$$(\mathfrak{m}_{x_0}/\mathfrak{m}^2_{x_0})^\vee \overset{\sim}{\longleftrightarrow} T_{x_0,\mathbb{A}^n_K} \overset{\sim}{\longleftrightarrow} \{D : \mathcal{O}_{\mathbb{A}^n_K,x_0} \to K \text{ satisfying (2), (3) and (4) above}\}. \tag{1}$$

Let now $\mathfrak{p} \subset K[x_1, \dots, x_n]$ be a prime ideal and let $X = V(\mathfrak{p}) \subset \mathbb{A}^n_K$ be the associated integral subscheme (note, we assume that $\mathfrak{p}$ is a prime only for intuition, but all we are going to say holds for any ideal). Let $x_0 \in X(K) \subset \mathbb{A}^n_K(K)$. We want to make sense of the tangent space $T_{x_0,X}$ of X at x_0. Intuitively, this must be a subspace of $T_{x_0,\mathbb{A}^n_K} \cong K^n$.

Assume for the sake of simplicity that $\mathfrak{p} = (g)$ so X is an hypersurface. For any $\vec{v}$ consider the line $\epsilon \mapsto x_0 + \epsilon\vec{v}$. Then since $g(x_0) = 0$ we surely have $g(x_0 + \epsilon\vec{v}) = O(\epsilon)$ (analysis notations). Then we can say that $\vec{v}$ is tangent to X at x_0 when we can get the better estimate $g(x_0 + \epsilon\vec{v}) = O(\epsilon^2)$ (for ϵ small). For example when $n = 2$ this simply says that the line $\{x_0 + \epsilon\vec{v}\}$ is the best linear approximation of $g = 0$ around x_0. Also, note that $g(x_0 + \epsilon\vec{v}) = O(\epsilon^2)$ if and only if $D_{x_0,\vec{v}}(g) = 0$. We then arrive at the following definition which matches our intuition from differential geometry:

Definition 8.20. Situation as above. We let

$$T_{x_0,X} = \{\vec{v} : D_{x_0,\vec{v}}(g) = 0 \text{ for every } g \in \mathfrak{p}\}.$$

Let now $\mathfrak{m}_{X,x_0} \subset K[x_1, \dots, x_n]/\mathfrak{p}$ be the maximal ideal corresponding to $x_0 \in X$, which is nothing but $\mathfrak{m}_{x_0}/\mathfrak{p}$.

Proposition 8.21. *The identification 1 induces a natural identification*

$$(\mathfrak{m}_{X,x_0}/\mathfrak{m}^2_{X,x_0})^\vee \cong T_{x_0,X}$$

where $T_{x_0,X}$ is defined above.

Proof. This is an easy but instructive exercise and it is left to the student. ☐

We thus arrive at the geoemtric interpretation of regularity:

Corollary 8.22. *Let $X \subset \mathbb{A}^n_K$ be an integral subvariety and let $x_0 \in X(K)$. Then X is regular at x_0 if and only if* $\dim(X) = \dim_K(T_{x_0,X})$.

Thus, a variety X is regular at $x_0 \in X(K)$ if and only if its tangent space has the same dimension of X. Recall that in Riemaniann geometry we can use the exponential functions to find a local homomorphism between some a small open ball of X at x_0 and some open ball of $T_{x_0,X}$ at 0 whenever X is a manifold. This means that when X is regular, X looks locally like $\mathbb{A}^n_K$. As I said at the beginning, this can be made precise in other ways also in the algebraic world.

Proposition 8.23 (Jacobian criterion). *Let $I \subset K[x_1, \dots, x_n]$ be an ideal generated by $f_1, \dots, f_r$ and let $x_0 \in V(I)$ be a K-rational point. Then $V(I)$ is regular at x_0 if and only if the Jacobian matrix $Jac(x_0) = \big(\frac{df_j}{dx_i}(x_0)\big)_{i,j}$ has rank $n - \dim_{x_0}(V(I))(= n - \dim(\mathcal{O}_{V(I),x_0})$.*

Proof. In fact,

$$T_{x_0,V(I)} = \{\vec{v} : D_{x_0,\vec{v}}(f) = 0 \text{ for every } f \in I\} = \{\vec{v} : \sum_i \frac{df_j}{dx_i} v_i = 0 \text{ for every } j = 1, \dots, r\}.$$

□

It is very easy using the Jacobian criterion to check that e.g. $(0,0)$ is the only singular point of the node $V(y^2 - x^2 - x^3)$ or of the cusp $V(y^2 - x^3)$. Also, it is very easy to check that $\mathbb{P}^n_K$ is regular at all its closed points, since it is covered by affine varieties isomorphis to $\mathbb{A}^n_K$. I state now the following important facts (the proof of the first is very hard)

1. Let X be a noetherian scheme. The X is regular if and only if it is regular at all its closed points;
2. Using the first fact and the fact that the determinant of a matrix is a continuous function, one also proves that for an affine variety X the set of its regular points is an open subset.

8.4 Miracle flatness

I want to conclude my part of the course by proving a weaker version of miracle flatness. This will also show how flatness, dimension and regularity mix together:

Theorem 8.24 (Miracle flatness, weak version). *Let $R \to S$ be a local morphism of noetherian local ring. Assume that*

1. *R, S are regular;*
2. $\dim(R) + \dim(S/\mathfrak{m}_R S) = \dim(S)$*;*
3. *The natural induced map $\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_S/\mathfrak{m}_S^2$ is injective.*

Then $R \to S$ is flat.

Note that $S/\mathfrak{m}_R S = S \otimes_R R/\mathfrak{m}_R$ and so $\mathrm{Spec}(S/\mathfrak{m}_R S)$ is nothing but the fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ over the closed point of $\mathrm{Spec}(R)$.

The full statement of miracle flatness is considerably stronger and only requires S to be Cohen-Macaulay (which is way weaker than regular, but which we do not have time to introduce in this course) and condition (3) does not appear at all. Condition (3) is needed to make induction work in our simplified version and it is closely related to the morphism being unramified. Other than this the proof follows the same logic.

Some preliminaries on flatness The only thing that we are going to assume is

Proposition 8.25 (Local criterion for flatness). *Let $R \to S$ be a local morphism of noetherian local rings and let k be the residue field of R. Let M be a finitely generated S-module. Then M is flat over R if and only if* $\mathrm{Tor}_1^R(k, M) = 0$.

Note that $\mathrm{Tor}_1^R(k, M) = 0$ if and only if $M \otimes_R \mathfrak{m} \to M$ is injective.

Lemma 8.26. *Let R be a ring, $I \subset R$ an ideal, M an R-module. Assume that*

1. *M/IM is flat R/I-module;*
2. $\mathrm{Tor}_1^R(R/I, M) = 0$.

Let N be any R-module annihilated by I. Then $\mathrm{Tor}_1^R(N, M) = 0$.

Proof. Since $IN = 0$ we can see N as an R/I-module. Find any presentation

$$0 \to K \to \bigoplus R/I \to N \to 0$$

and note that all the modules above are R/I-modules. Now apply $- \otimes_R M$ to the sequence above and use that Tor commutes with direct sum and that $\mathrm{Tor}_1^R(R/I, M) = 0$ by assumption to obtain a sequence

$$0 \to \mathrm{Tor}_1^R(N, M) \xrightarrow{\alpha} K \otimes_R M \to (\bigoplus R/I) \otimes_R M \to N \otimes_R M \to 0$$

but $K \otimes_R M = K \otimes_{R/I} M/IM$ since K is a R/I-module and similarly for $\bigoplus R/I$ and N. Since M/I is flat R/I-module we have the exact sequence

$$0 \to K \otimes_{R/I} M/IM \to (\bigoplus R/I) \otimes_{R/I} M/IM \to N \otimes_{R/I} M/IM \to 0$$

hence α must be injective, proving the result. □

We thus have the following useful variant of the local criterion for flatness:

Proposition 8.27. *Let $R \to S$ be a local morphism of noetherian local rings and let M be a finitely generated S-module. Assume that*

1. *M/IM is flat R/I-module;*
2. *$\operatorname{Tor}_1^R(R/I, M) = 0$.*

Then M is flat R-module.

Proof. Apply the previous result to $N = R/\mathfrak{m} = k$, which is annihilated by I, and deduce that $\operatorname{Tor}_1^R(k, M) = 0$. Then use the local criterion for flatness. □

Some facts about regular local rings Many results about local rings are proven using induction on the dimension via the following

Proposition 8.28. *Let $(R, \mathfrak{m})$ be a regular local ring and let $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then R/f is also a regular local ring (of dimension $\dim(R) - 1$).*

Proof. Note that $S := R/f$ is local with maximal ideal $\mathfrak{n} = \mathfrak{m}/f$ and same residue field k. Since $f \notin \mathfrak{m}^2$ we must have $\dim_k(\mathfrak{n}/\mathfrak{n}^2) = \dim_k(\mathfrak{m}/\mathfrak{m}^2) - 1 = \dim(R) - 1$ since R is regular. By the results in the previous lecture we also have

$$\dim(R) - 1 \leq \dim(S) \leq \dim_k(\mathfrak{n}/\mathfrak{n}^2) = \dim(R) - 1$$

hence $\dim(S) = \dim(R) - 1$ and $\dim(S) = \dim_k(\mathfrak{n}/\mathfrak{n}^2)$ as required. □

Regularity puts a lot of constrains on R. In fact, if $(R, \mathfrak{m})$ is a regular local ring then R is necessarily an integrally closed domain. We prove that R is a domain, the fact that it is also integrally closed is considerably harder:

Theorem 8.29. *Let $(R, \mathfrak{m})$ be a regular local ring. Then R is an integral domain.*

Proof. It is enough to show that (0) is the only minimal prime of R. In fact, it is enough to show that any minimal prime $\mathfrak{p}$ of R satisfying $\dim(R/\mathfrak{p}) = \dim(R)$ must be 0 (in which case, since 0 is a prime, R is a domain. Also note that $\dim(R) = \max_{\mathfrak{p}}(\dim(R/\mathfrak{p}))$ where p ranges among the – finitely many – minimal primes of R). We prove this by induction on $\dim(R)$. If $\dim(R) = 0$

then R is artinian and $\mathfrak{m}/\mathfrak{m}^2 = 0$ by regularity hence $\mathfrak{m} = \mathfrak{m}^2$ hence $\mathfrak{m} = 0$ by Nakayama. So R is a field. Assume now $\dim(R) > 0$ and let $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. We know by the previous result that R/f is regular of dimension $\dim(R) - 1$. Hence R/f is an integral domain by induction.

Let now $\mathfrak{p} \subset R$ be a minimal prime such that $\dim(R) = \dim(R/\mathfrak{p})$. Write $S = R/\mathfrak{p}$ which is a local domain with maximal ideal $\mathfrak{n} = \mathfrak{m}/\mathfrak{p}$ and let $g \in S$ be the image of f. Now we have

$$\dim(R) - 1 = \dim(S) - 1 \leq \dim(S/g) \leq \dim_k(\mathfrak{n}/\mathfrak{n}^2) = \dim(R) - 1$$

where the first equality comes from the assumption. Hence also S/g is regular of dimension $\dim(R) - 1$. So both R/f and S/g are regular of dimension $\dim(R) - 1$ and by induction they are integral domains. But $R/f \twoheadrightarrow S/g$ hence $R/f = S/g$ (if R, S are noetherian integral domains and S is a quotient of R, if $\dim(R) = \dim(S)$ then $R = S$). Hence $(f) = \mathfrak{p} + (f)$ and so $\mathfrak{p} \subset (f)$. So for any $u \in \mathfrak{p}$ there is $v \in R$ such that $u = vf$. Now note that $f \notin \mathfrak{p}$ for $\dim(V(f)) = \dim(R/f) = \dim(R) - 1$ and $\dim(R/\mathfrak{p}) = \dim(R)$ but if $f \in \mathfrak{p}$ then $V(f) \subset V(\mathfrak{p})$ which leads to a contradiction. Since $\mathfrak{p}$ is prime we must have $v \in \mathfrak{p}$ hence $\mathfrak{p} \subset f\mathfrak{p} \subset \mathfrak{m}\mathfrak{p}$ and finally $\mathfrak{p} = 0$ by Nakayama. □

We now prove our simplified version of miracle flatness:

Proof of Theorem 8.24. Before proving the result let us notice that $R \to S$ is necessarily injective under the conditions (1) and (2). Let K be the kernel. Since R is a domain because regular, $K = 0$ if and only if the induced map $\phi : \overline{\operatorname{Spec}(S)} \to \operatorname{Spec}(R)$ is dominant. But if it were not dominant then $\dim(S) \leq \dim(\overline{\operatorname{Im}(\phi)}) + \dim(S/\mathfrak{m}_R S) < \dim(R) + \dim(S/\mathfrak{m}_R S)$ where the strict inequality $\dim(\operatorname{Im}(\phi)) < \dim(R)$ is true because R is an integral domain. This contradicts condition (2) and so $K = 0$.

We now prove the theorem by induction on $\dim(R)$. So if $\dim(R) = 0$ then R is a field and everything is flat over R. So assume $\dim(R) > 0$. Since $R \to S$ is a local morphism $\mathfrak{m}_S^c = \mathfrak{m}_R$. Pick now any $x \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2$ and note that the image of x in S does not belong to $\mathfrak{m}_S^2$ by assumption. Henceforth both $R/(x)$ and $S/(x)$ are regular of dimension $\dim(R) - 1$ and $\dim(S) - 1$ respectively. On the other hand $S/\mathfrak{m}_R S = (S/x)/\mathfrak{m}_R(S/x)$ and so condition (2) of the theorem is also satisfied for the morphism $R/x \to S/x$. Clearly also condition (3) is satisfied. Thus S/x is flat over R/x by induction. We now want to apply Proposition 8.27 with $I = (x)$ and $M = S$ to conclude. So we only need to check that $\operatorname{Tor}_1^R(R/x, S) = 0$. But we know that regular local rings are integral domains, so the sequence

$$0 \to R \xrightarrow{x} R \to R/x \to 0$$

is exact; now tensoring with S we get

$$0 = \operatorname{Tor}_1^R(R, S) \to \operatorname{Tor}_1^R(R/x, S) \to S \xrightarrow{x} S \to S/x \to 0$$

and since also S is regular, x is not a zero divisor on S either because $x \neq 0$ in S, and so $\operatorname{Tor}_1^R(R/x, S) = 0$. This concludes the proof. □