5 Letctures VII/ VIII

5.1 Some global properties of schemes

Schemes are very general objects, whereas varieties (which are in the end our object of interest) are very particular in respect, since they are obtained by glueing the Spec of finitely generated, integral K-algebras. So already to describe varieties among schemes we need a handful of properties.

Definition 5.1. Let X be a scheme. Then X is

1. *Connected* if the topological space of X is connected.

2. *Irreducible* if the topological space of X is irreducible;

3. *Reduced* if for every affine open $U = \mathrm{Spec}(A) \subset X$ we have that A has no nilpotents;

4. *Integral* if it is reduced and irreducible.

Let us characterise these properties:

Proposition 5.2. *Let X be a scheme, then*

1. *X is irreducible iff for every open affine* $\mathrm{Spec}(A) \subset X$ *the nilradical of A is prime;*

2. *X is reduced iff all its local rings have no nilpotents iff for every open affine* $\mathrm{Spec}(A) \subset X$ *the nilradical of A is zero;*

3. *X is integral iff for every open affine* $\mathrm{Spec}(A) \subset X$ *the ring A is a domain;*

Proof.

1. Suppose that X is irreducible and let $U \subset X$ be an open subset. Then also U is irreducible. But so if $U = \mathrm{Spec}(A)$ this means that A has a minimal prime ideal, which necessarily coincides with the nilradical. On the other hand, assume that for any such open the nilradical of A is not prime. Thus there are at least two different minimal primes in A, showing that $\mathrm{Spec}(A)$ cannot be irreducible.

2. Obvious, since nilpotent elements belong to every prime ideal, hence they inject into every localization.

3. Follows from the previous two.

□

Remark 5.3. Let X be an integral scheme. Then X has a unique generic point $\eta \in X$, i.e., a point such that $\bar{\eta} = X$. Let $U = \mathrm{Spec}(A) \subset X$ be any affine open. Since A is an integral domain $(0) \subset A$ is a prime ideal and η is just the image of (0) under the inclusion. This is easily well-defined. Its function field is $\mathrm{Frac}(A)$ which is independent on the chosen A.

Now, noetherianity and quasi-compactness.

Definition 5.4. A scheme X is quasi-compact if its topological space is quasi-compact. A scheme X is locally noetherian if we can cover it with affine opens $U_i = \mathrm{Spec}(A_i)$ where each A_i is a noetherian ring. It is noetherian if it is locally noetherian and quasi-compact (so we can find a finite number of such covers).

Note that many times in scheme theory we will encounter situations where a property is spelled out like 'there exists an affine open cover such that...'. In many situations – if not all – if this is true for one affine open cover then it is true for every affine open cover. Let us make an example:

Proposition 5.5. *Assume that X is locally noetherian. Then for every open affine* $U = \mathrm{Spec}(A) \subset X$ *we have that A is noetherian ring.*

Proof. We know that we can cover X with $U_i = \mathrm{Spec}(A_i)$ where each A_i is a notherian ring. So we can cover U with $U_i \cap U$. Since U is affine it is quasi-compact, hence we can find finitely many such opens. Moreover, for $a_i \in A_i$ note that A_{i,a_i} is also noetherian, and that we can cover each $U \cap U_i$ with finitely principal open of $\mathrm{Spec}(A_{i,a_i})$ each one noetherian. Thus we are reduced to prove the following: let $X = \mathrm{Spec}(A)$ be an affine scheme which can be covered by affine opens which are the Spec of noetherian rings. Then A is notherian.

Let $U = \mathrm{Spec}(B) \subset X$ be open with B noetherian. Then there must be some $f \in A$ such that $D(f) \subset U$ hence the inclusions $D(f) \subset U \subset X$ give maps of rings $A \to B \to A_f$. But then $B_f = A_f$ necessarily, hence also A_f is notherian.

Thus we are reduced to the following: let A be a ring and let $f_1, \cdots, f_n \in A$ such that $1 = (f_1, \cdots, f_n)$. Assume that each $A_i = A_{f_i}$ is notherian. Then A is notherian.

The proof rests now on this simple claim: in the situation above, let $\phi_i : A \to A_i$ be the localization map. Then for every ideal $I \subset A$ we have an equality

$$I = \bigcup_i \phi_i^{-1}(IA_i).$$

One inclusion is obvious. For the second, pick x in the intersection. Then for every i we can write $x = i_i/f_i^{n_i}$ for some $i_i \in I$. Hence we can find $m > n > 0$

such that for every i we have

$$f_i^m(i_i/f_i^n - x) = i_i f_i^{m-n} - f_i^m x = 0.$$

Now we can write $1 = \sum_i r_i f_i^m$ hence

$$x = (\sum_i r_i f_i^m)x = \sum_i r_i f_i^m x \in I$$

which proves the claim. Now, for every increasing sequence of ideals $I_1 \subset I_2 \subset \cdots$ we obtain for every i an increasing sequence $I_1 A_i \subset I_2 A_i \subset \cdots$ which must stabilize since A_i is notherian. Hence there is some $n >> 0$ such that $I_k A_i = I_{k+1} A_i$ for every $k \geq n$ and for every i. This implies that $I_k = I_{k+1}$ for every $k \geq n$ by the claim. □

In this way, we can say that Noetherianity is a local property of rings. In the previous lecture, we distinguished two types of properties for schemes: absolute properties (like the one above, which do not refer to a structure morphism) and relative properties, which pertain to morphisms rather than to schemes themselves. Before introducing the latter, we define one of the most important tools in algebraic geometry: fibre products.

5.2 Fibred products

Let $\mathscr{C}$ be any category and let

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & S \end{array}$$

be a diagram of morphisms of $\mathscr{C}$.

Definition 5.6. The fibred product $X \times_S Y$, if it exists, is the universal object in $\mathscr{C}$ which makes the following diagram commute

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\pi_Y} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow \\ X & \longrightarrow & S \end{array}$$

By the universal property, if the fibred product exists, it is unique up to unique isomorphism.

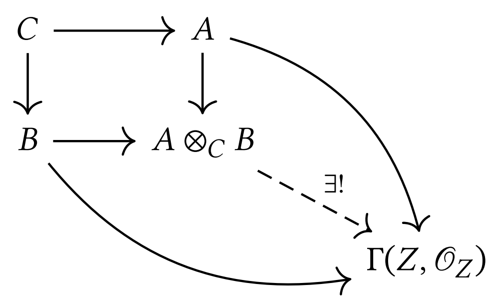
Example 5.7. Let $\mathscr{C}$ be the category of sets. Then

1. If $S = \{*\}$ the fibred product is the usual product;
2. If $s \in S$ and $f : X \to S$ then $X \times_S \{s\} = X_s = f^{-1}(s)$.
3. If both $X, Y \subset S$ then $X \times_S Y = X \cap Y$.

Theorem 5.8. *Let X, Y be schemes over a base scheme S. Then the fibred product $X \times_S Y$ exists.*

Proof. The proof follows several steps.

- Step 1: all X, Y, S are affine. Write $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(C)$ so that we have maps of rings $C \to A$ and $C \to B$. We claim that $\mathrm{Spec}(A \otimes_C B)$ together with the natural maps $A, B \to A \otimes_C B$ is the fibred product $X \times_S Y$. We only need to check the universal property. So let Z be any scheme with maps $Z \to X, Y$ making the diagram commute. We know that this is equivalently given by ring maps $A, B \to \Gamma(Z, \mathcal{O}_Z)$ which restrict to the same map on C. Hence, by the universal property of tensor products, we can find a unique map of rings $A \otimes_C B \to \Gamma(Z, \mathcal{O}_Z)$ making the diagram commute



This shows that $\mathrm{Spec}(A \otimes_C B)$ satisfies the universal property.

- Step 2: restrictions. If $X \times_S Y$ exists and $U \subset X$ is open, then $\pi_X^{-1}(U) \subset X \times_S Y$ is the fibred product $U \times_S Y$. This is easy using the universal property.

- Step 3: Glueing. Assume that $U_i \subset X$ is an open cover such that $U_i \times_S Y$ exists for every i. Then also $X \times_S Y$ exists. In fact, let $U_{ij} = U_i \cap U_j$. By the previous point $U_{ij} \times_S Y$ exists and it is in fact an open subscheme of $U_i \times_S Y$ as well as of $U_j \times_S Y$. So now we can glue the schemes $U_i \times_S Y$ along the open subschemes $U_{ij} \times_S Y$ to obtain a new scheme

$$\Big(\bigsqcup_i U_i \times_S Y\Big)_{/\sim}.$$

We need to show that this glued scheme is the required fibred product. So let $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$ be maps making the usual diagram commute and let $Z_i = f^{-1}(U_i)$. We get maps $f_i : Z_i \to U_i$ and $g_i : Z_i \to Y$ (the restriction of g to the open Z_i) making again the usual diagram commute. Hence, by the universal property for U_i, we get a unique map $Z_i \to U_i \times_S Y$ for every i. By the same universal property, these maps agree on the intersections $Z_{ij} = Z_i \cap Z_j$ in the sense that on such intersection they factor through the open immersions $U_{ij} \times_S Y \hookrightarrow U_i \times_S Y, U_j \times_S Y$ and match over the identifications. Hence they glue to a unique map $Z \to \left(\bigsqcup_i U_i \times_S Y\right)_{/\sim}$, which shows that the glued scheme represents the fibred product.

- Step 4: conclusion. All in all, this shows that $X \times_S Y$ exists whenever S is affine. Assume finally that S is not affine and cover it by affine opens $S_i \subset S$. Let X_i be the preimage of S_i in X and similarly for Y_i. Then $X_i \times_{S_i} Y_i$ exists for every i. It is easy to check that

$$X_i \times_{S_i} Y_i \;\cong\; X_i \times_S Y,$$

hence we use the previous point to conclude.

□

Example 5.9.
- Fibres of a morphism. Let $f : X \to Y$ be a morphism and let $y \in Y$. Let $\kappa(y)$ be the residue field of y, so we have a natural map of schemes $\mathrm{Spec}(\kappa(y)) \to Y$. We define

$$X_y \;=\; X \times_Y \mathrm{Spec}(\kappa(y))$$

and we call it the fibre of X over y. For example, assume that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ and f is given by $A \to B$. For a prime $\mathfrak{p} \subset A$ corresponding to y we then have $X_y = \mathrm{Spec}\left(B \otimes_A \kappa(\mathfrak{p})\right)$. What are the primes of $B \otimes_A \kappa(\mathfrak{p})$? Consider the following commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B_{\mathfrak{p}} & \longrightarrow & A \otimes_B B_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ B_{\mathfrak{p}}/\mathfrak{p} = \kappa(\mathfrak{p}) & \longrightarrow & A \otimes_B \kappa(\mathfrak{p}) \end{array}$$

Since the last horizontal map is necessarily an injection if the tensor is not zero, the commutativity of the diagram shows that the points in X_y

correspond precisely to the points of X mapped to y, i.e., to the prime $\mathfrak{q} \subset A$ such that $\mathfrak{q}^c = \mathfrak{p}$.

- Let R be any integral ring and consider $S = R[x_1, \cdots, x_n]/I$ where I is an ideal such that $I \cap R = (0)$. We thus get a morphism of schemes $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. Let $\mathfrak{p} \subset R$ be a prime ideal and let $I_{\mathfrak{p}}$ be the image of $I \otimes_R \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p})[x_1, \cdots, x_n]$ (note that this is not in general injective—see later flatness). Then for every prime $\mathfrak{p} \subset R$ the fibre is $\mathrm{Spec}\big(\kappa(\mathfrak{p})[x_1, \cdots, x_n]/I_{\mathfrak{p}}\big)$.

- Let now $R = K[t]$ and let $S = R[x, y]/(a(t)x + b(t)y - c)$ for some $a(t), b(t) \in R$ and $c \in K$. We can consider the natural map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. We can consider this as a family of lines. For any maximal ideal (assume K algebraically closed) corresponding to some $\alpha \in K$ the corresponding fibre is the line $K[x, y]/\big(a(\alpha)x + b(\alpha)y - c\big)$. For instance, note that if $c \neq 0$ and both $a(\alpha) = b(\alpha) = 0$ then the fibre is empty. It is otherwise a line passing through c;

- One can make many examples of the situation above. For example, $S = R[x, y]/(P_t(x, y))$ where $P_t(x, y) \in K[t, x, y]$. Then the fibre for the closed point $t = \alpha$ is nothing but $K[x, y]/\big(P_\alpha(x, y)\big)$. So we get a 'family of varieties' $t_0 \mapsto V(P(x, y, t_0))$.

- Base-extension: If $X \to S$ is an S-scheme and $S' \to S$ is a map of schemes, we call $X_{S'} = X \times_S S' \to S'$ the *base-extension* of X to S'. For example, let
$$X = \mathrm{Spec}\big(\mathbb{Q}[x, y]/(x^2 + 3y^2)\big).$$
Then X is an integral scheme. Let $\mathbb{Q} \subset \mathbb{Q}(\sqrt{-3})$ and consider the fibre product
$$X \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}\big(\mathbb{Q}(\sqrt{-3})\big) = \mathrm{Spec}\Big(\mathbb{Q}(\sqrt{-3})[x, y]/\big((x+\sqrt{-3}y)(x-\sqrt{-3}y)\big)\Big),$$
which is not irreducible anymore. In fact, properties like reduced/irreducible are not stable under base extension.

5.3 Some finiteness properties of morphisms

The most basic finiteness properties of morphisms are:

Definition 5.10. Let $f : X \to Y$ be a morphism of schemes. We say that f is

1. Locally of finite type: if there is an open cover $U_i = \mathrm{Spec}(A_i) \subset Y$ such that $f^{-1}(U_i) = V_i \subset X$ can be covered by affine opens $V_{i,j} = \mathrm{Spec}(B_{ij})$ with B_{ij} a finitely generated A_i-algebra for every i, j.

2. <u>*Of finite type*</u>: as above, and for each i we can choose finitely many $V_{i,j}$ (automatic if X is quasi-compact).

3. <u>*Finite*</u>: if there is an affine open cover $U_i = \mathrm{Spec}(A_i) \subset Y$ such that $f^{-1}(U_i)$ is affine, say $f^{-1}(U_i) = \mathrm{Spec}(B_i)$, and each B_i is a finitely generated A_i-module.

These properties are defined by the existence of an open cover of the base with the stated behavior, so they are *local on the base*. Closed immersions are easily seen to be finite morphisms. We have:

Proposition 5.11. *Let $f : X \to Y$ be as above.*

1. *If f is locally of finite type, then for* every *affine open $U = \mathrm{Spec}(A) \subset Y$ and* every *affine open* $\mathrm{Spec}(B) \subset f^{-1}(U)$*, the ring B is a finitely generated A-algebra.*

2. *If f is finite, then for* every *affine open $U \subset Y$ the preimage $f^{-1}(U)$ is affine, and the induced map of rings is integral.*

In general, if $f : X \to Y$ is a morphism and $U = \mathrm{Spec}(A) \subset Y$ is affine, there is no reason that $f^{-1}(U)$ is affine. When this happens, we say that f is *affine*. Thus finite morphisms are, in particular, affine.

To prove results of this kind, it is useful to have the following lemma.

Lemma 5.12. *Let A, B be rings, and suppose there exists a scheme U together with open immersions $U \hookrightarrow \mathrm{Spec}(A)$ and $U \hookrightarrow \mathrm{Spec}(B)$. Then there are $a \in A$ and $b \in B$ such that $A_a \cong B_b$.*

Proof. We may assume U is affine. It suffices to find $a \in A$ and $b \in B$ with $D(a), D(b) \subset U$ and $D(a) = D(b)$ as topological spaces. In that case the affine schemes $D(a)$ and $D(b)$ are isomorphic (all maps are open immersions), which yields the claim.

Write $U = \mathrm{Spec}(C)$. Let the maps induced by the open immersions be $A \xrightarrow{\phi} C$ and $B \xrightarrow{\psi} C$. Pick $a \in A$ with $D(a) \subset U$. We claim $A_a \cong C_{\phi(a)}$. Indeed, since $D(a) \subset U$, the map $A \to A_a$ factors as $A \to C \to A_a$, so by the universal property of localization we obtain inverse maps $A_a \leftrightarrows C_{\phi(a)}$.

Similarly, choose $b \in B$ with $D(b) \subset U$; then $B_b \cong C_{\psi(b)}$. Let $c = \phi(a)\psi(b) \in C$. We can thus find $\tilde{a} \in A$ and $\tilde{b} \in B$ whose images in $C_{\phi(a)}$ and $C_{\psi(b)}$ map to c up to a unit. Localizing then yields $A_{\tilde{a}} \cong C_c \cong B_{\tilde{b}}$. □

Proof of the previous proposition. We prove (1). Split the statement into two parts:

1. Step 1: For every affine open $U = \mathrm{Spec}(A) \subset Y$, the preimage $f^{-1}(U)$ is covered by affine opens $\mathrm{Spec}(B_k)$ with each B_k a finitely generated A-algebra. This is known for some affine open cover $U_i = \mathrm{Spec}(A_i)$ of Y. Set $V_i = U \cap U_i$, which covers U; note V_i need not be affine, and since U is quasi-compact we may assume only finitely many indices i occur. By the lemma, there exist finitely many $a' \in A_i$ and $a \in A$ with $A_{i,a'} \cong A_a$ such that the basic opens $\mathrm{Spec}(A_a) = \mathrm{Spec}(A_{i,a'})$ cover V_i. By assumption, $f^{-1}(\mathrm{Spec}(A_i))$ is covered by $\mathrm{Spec}(B_{ij})$, so $f^{-1}(\mathrm{Spec}(A_{i,a'})) = f^{-1}(\mathrm{Spec}(A_a))$ is covered by the $\mathrm{Spec}(B_{ij,a'})$. Each B_{ij} is a finitely generated A_i-algebra, hence $B_{ij,a'}$ is a finitely generated $A_{i,a'} \cong A_a$-algebra. Since A_a is a finitely generated A-algebra, each $B_{ij,a'}$ is a finitely generated A-algebra. All in all, this shows that $f^{-1}(U)$ is covered by opens of the form $\mathrm{Spec}(B_{ij,a'})$ where only finitely many i and $a' \in A_i$ occur and such that each $B_{ij,a'}$ is a finitely generated A-algebra.

2. Step 2: for any affine $\mathrm{Spec}(B) \subset f^{-1}(\mathrm{Spec}(A))$ we have that B is a finitely generated A-algebra. From (1) and the lemma, choose $b_1, \dots, b_n \in B$ with $(b_1, \dots, b_n) = 1$ and such that each B_{b_i} is a finitely generated A-algebra. Pick a relation $1 = \sum c_i b_i$ in B and a finite set $\mathscr{C} \subset B$ that generates each B_{b_i} as an A-algebra and contains $\{c_i\}$ and $\{b_i\}$. Let $B' \subset B$ be the A-subalgebra generated by $\mathscr{G}$. Then $c_i, b_i \in B'$, so the b_i generate the unit ideal of B' too. Viewing B as a B'-module, we have $B'_{b_i} \cong B_{b_i}$ since $\mathscr{C}$ generates each B_{b_i}; since the $D(b_i)$ cover $\mathrm{Spec}(B')$, it follows that $B' = B$. Hence B is finitely generated by $\mathscr{C}$ over A.

The same argument works for finite morphisms; we omit the details. □

Definition 5.13. A property $\mathscr{P}$ of morphisms of schemes is *stable under base change* if whenever $f : X \to Y$ has $\mathscr{P}$ and $Y' \to Y$ is any morphism, the base change $X_{Y'} \to Y'$ also has $\mathscr{P}$.

Clearly, being of finite type or finite is stable under base change. Let $f : X \to Y$ be of finite type. Then for every $y \in Y$, the fiber X_y is a scheme over $\kappa(y)$ covered by finitely many finitely generated $\kappa(y)$-algebras, so X_y is "almost a variety" over $\kappa(y)$. The slogan is: morphisms of finite type = families of varieties.

Example 5.14.
1. Let $Q(x, y) = y^2 - x^3 - 7{\cdot}5\, x^2 + 7{\cdot}11$ and $R = \mathbb{Z}[x, y]/(Q(x, y))$. The generic fiber is $R_{\mathbb{Q}} = \mathbb{Q}[x, y]/(Q(x, y))$, an affine elliptic curve. For a prime p with residue field $\mathbb{F}_p$, the fiber over p is $\mathbb{F}_p[x, y]/(Q_p(x, y))$, where Q_p is the reduction of Q modulo p. For $p = 7$ we get $Q_p(x, y) = y^2 - x^3$, the cusp (additive reduction). For $p = 11$ we get $Q_{11}(x, y) =$

$y^2 - x^3 - 2x^2$, the node (multiplicative reduction). For all other primes we obtain an affine elliptic curve.

2. Let us look again at the example from the previous lecture. Let K be a field and $P(x,y,t) \in K[x,y,t]$. The map $K[t] \to R = K[x,y,t]/(P)$ is of finite type. For $t_0 \in K$, the fiber over $t - t_0$ is

$$\operatorname{Spec}\left(K[x,y]/(P(x,y,t_0))\right),$$

so set-theoretically we get a family $t_0 \mapsto V(P(x,y,t_0))$ of curves. Unwanted behaviors can occur: if some t_0 satisfies $P(x,y,t_0) = 0$, the fiber is the whole plane $\mathbb{A}^2_K$ (dimension jump). Degrees can also drop, e.g. for $P(x,y,t) = tx^3 + y + 1$ the fiber at $t = 0$ is a line, while generically it is a cubic.

This shows that maps of schemes do not always yields continuous families of varieties parametrized by the points of the base. To avoid such jumps and obtain continuity, one assumes flatness (an insight of Serre).

Definition 5.15. A ring map $A \to B$ is *flat* if B is a flat A-module, equivalently if $B_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}^c}$-module for every prime $\mathfrak{p}$ of B. A morphism of schemes is flat if for every Zariski open of the base, its preimage can be covered by affine opens that are flat over it. Equivalently, if for every $x \in X$ the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

We will study flatness in more depth later. For now, here is a key application.

5.4 Finite flat morphisms have fibers of the same cardinality

Let Y be an integral scheme and $f : X \to Y$ finite and flat. Let $x \in X$ with $y = f(x)$. Choose an affine open $U = \operatorname{Spec}(A) \subset Y$ containing y. Then $f^{-1}(U) = \operatorname{Spec}(B)$ is affine, $A \to B$ is integral, and B is a finitely generated A-module. If y corresponds to $\mathfrak{p} \subset A$, the fiber over y is $\operatorname{Spec}\left(B \otimes_A \kappa(\mathfrak{p})\right)$, a finite-dimensional $\kappa(\mathfrak{p})$-vector space, and we set

$$|X_{f(x)}| = \dim_{\kappa(f(x))}\left(B \otimes_A \kappa(f(x))\right).$$

Theorem 5.16. *If $f : X \to Y$ is finite and flat with Y integral, then $|X_{f(x)}|$ is independent of $x \in X$.*

Proof. Let $x \in X$, $y = f(x)$, and choose $U = \operatorname{Spec}(A) \subset Y$ with $y \in U$, writing $f^{-1}(U) = \operatorname{Spec}(B)$. Then x corresponds to a prime $\mathfrak{p} \subset B$ and y to $\mathfrak{p}^c \subset A$. Flatness implies $A_{\mathfrak{p}^c} \to B_{\mathfrak{p}^c}$ is flat.

Let now $(R, \mathfrak{m})$ be a local domain and let M be a finitely generated flat R-module. Note that M is torsion free (since R is a domain). Let k be the residue field of R and F be its fraction field. Then $M_k = M \otimes k$ is a finite dimensional k-vector space; fix a basis $\tilde{x}_1 \dots \tilde{x}_r$ of M_k and lift it to elements $x_i \in M$. By Nakayama's lemma then $x_1, \dots, x_r$ generate M as well. We need to show that they also form a basis of $M_F = M \otimes F$. But they generate M_F as an F-vector space, so we get $\dim_F(M_F) \le \dim_k(M_k)$ and we need to prove equality. If $x_1, \dots, x_r$ were not linearly independent in M_F then there are some $a_i \in F$ not all zero such that $\sum r_i x_i = 0$. By clearing denominators we can assume that $r_i \in R$ and $\sum r_i x_i = 0$ in M. But then necessarily $r_i \in \mathfrak{m}$ for every i, for otherwise we would get a non-trivial relation in M_k. Pick the minimal n such that all $r_i \in \mathfrak{m}^n$ but there is some i with $r_i \notin \mathfrak{m}^{n+1}$. We can assume $i = 1$. So r_1 is non-zero in $\mathfrak{m}^n/\mathfrak{m}^{n+1}$, which is a finitely dimensional k-vector space. Now, by flatness the natural map $\mathfrak{m}^n \otimes M \to M \otimes R = M$ is injective. Consider the element $\sum_i r_i \otimes x_i \in \mathfrak{m}^n \otimes M$, which is sent to zero in M. Then it must be zero already in the tensor product. Pick any A-linear map $f : \mathfrak{m}^n/\mathfrak{m}^{n+1} \to k$ such that $f(r_1) \neq 0$ (note that A-linear in this context is equivalent to k-linear). So we get a map

$$\mathfrak{m}^n \otimes M \to (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes M \xrightarrow{f \otimes \mathrm{Id}_M} k \otimes M = M_k$$

which sends $0 = \sum_i r_i \otimes x_i \mapsto \sum_i f(r_i)\tilde{x}_i$. But then $f(r_i) = 0$ for every i necessarily, which is a contradiction. This shows that $\dim_F(M_F) = \dim_k(M_k)$, as required.

Now we conclude the proof. Let $A \subset B$ be an integral extension of rings; since in our case A is a domain, then also B is a domain. Pick a prime $\mathfrak{p} \subset B$ and consider the induced integral extension $A_{\mathfrak{p}^c} \subset B_{\mathfrak{p}^c}$. By assumption $A_{\mathfrak{p}^c}$ is a local domain and $B_{\mathfrak{p}^c}$ is a finitely generated flat $A_{\mathfrak{p}^c}$-module. So we use the previous computation to get the equality

$$\dim_{\kappa(\mathfrak{p}^c)} \kappa(\mathfrak{p}^c) \otimes_{A_{\mathfrak{p}^c}} B_{\mathfrak{p}^c} = \dim_{\mathrm{Frac(A)}} B \otimes_A \mathrm{Frac(A)}$$

but clearly $\kappa(\mathfrak{p}^c) \otimes_{A_{\mathfrak{p}^c}} B_{\mathfrak{p}^c} = \kappa(\mathfrak{p}^c) \otimes_A B$ which proves the result. □

This illustrates again why nilpotents matter. Consider K a field and $n \ge 1$. The map $\mathbb{A}^1_K \to \mathbb{A}^1_K$ induced by $K[t] \xrightarrow{t \mapsto t^n} K[t]$ is finite and free (hence flat). The fiber over $(t - t_0)$ is $K[t]/(t^n - t_0)$. If $\operatorname{char} K = p > 0$ and $n = p$, then for perfect K we can choose t_0' with $(t_0')^p = t_0$, and $K[t]/(t^p - t_0) \cong K[t]/(t - t_0')^p$ is nonreduced with a single point, but $\dim_K K[t]/(t - t_0')^p = p$ accounts for the "missing'' points; if K is not perfect and t_0 is not a p-th power, then $K[t]/(t^p - t_0)$ is reduced but not geometrically reduced; if $(n, p) = 1$, then each fiber over

$t_0 \neq 0$ has n distinct points, while the fiber over $t_0 = 0$ is the single point 0 with ring $K[t]/(t^n)$ recording the lost points as before. This shows the concept of multiplicity.

Examples (Glimpses of ramification). Let A be a domain, $K = \mathrm{Frac}(A)$, and L/K a finite extension. Let B be the integral closure of A in L, so $A \subset B$ is integral. In general, $A \subset B$ need not be flat. For example, let $A = K[x,y]/(x^2 - y^3)$ be the affine cusp and $L = \mathrm{Frac}(A)$. Then the integral closure is $K[t]$ with $t^2 = y$ and $t^3 = x$ (indeed $t = x/y$), so $A \subset K[t]$ is integral. Let us show that it is not flat. We have $A_x \to K[t]_x = K[t]_t$ an isomorphism, so each fiber of $f : \mathrm{Spec}(A) \to \mathbb{A}^1_K$ has cardinality 1 over $D(t)$. To see nonflatness, consider $(t)^c = (x,y)$ and $A_{(x,y)} \to K[t]_{(t)}$. The maximal ideal of $A_{(x,y)}$ is not principal, so $\dim_K (x,y)/(x,y)^2 \geq 2$, whereas the maximal ideal of $K[t]_{(t)}$ is generated by t, so $\dim_K (t)/(t)^2 = 1$. Thus the map sends (x,y) into $(t)^2$, i.e. the morphism is *ramified* at (x,y) (this is in fact the definition of unramified morphism together with separability of residue extensions). One computes

$$\dim_K\big(A/(x,y) \otimes_A K[t]\big) = \dim_K K[t]/(t^2,t^3) = \dim_K K[t]/(t^2) = 2,$$

whereas over the function field the dimension is 1, so here ramification kills flatness.

Ramification need not obstruct flatness in general. Let $A_\mathfrak{p} \to B_\mathfrak{p}$ be the induced map (by going up, every prime of A is the contraction of some prime of B). If A is noetherian, $B_\mathfrak{p}$ has finitely many minimal primes $\mathfrak{q}_1, \dots, \mathfrak{q}_k$. Some $A_\mathfrak{p} \to B_{\mathfrak{q}_i}$ may be ramified, yet $A_\mathfrak{p} \to B_\mathfrak{p}$ can still be flat. For instance, if A is a Dedekind domain, then B is automatically flat: B is torsion-free and finitely generated over a Dedekind domain, whose local rings are PIDs, hence B is locally free and therefore flat (by the classification of finitely generated modules over PID;s). The miracle flatness theorem generalizes this to higher-dimensional regular rings.

For example, $K[t]$ is a Dedekind domain. Let $K(t) \subset L$ be a finite extension generated by $\alpha \in L$ with minimal polynomial $P(x) = x^n + a_{n-1}(t)x^{n-1} + \cdots + a_0(t) \in K(t)[x]$. Let B be as above. Choose a monic $b \in K[t]$ minimal (for divisibility) such that $bP(x) \in K[t,x]$. For a maximal ideal $\mathfrak{m} = (t - t_0)$, either $b(t_0) \neq 0$ or $b(t_0) = 0$. In the first case, α satisfies a monic equation over $K[t]_\mathfrak{m}$ because b is invertible in this ring. Since integral closure commutes with localization we deduce that α (although not necessarily integral over A) belongs to the image of $B \otimes_A A_\mathfrak{m}$ in $B_\mathfrak{m}$. Hence $B_\mathfrak{m}$ is generated by $1, \alpha, \dots, \alpha^{n-1}$ where $n = \deg(P)$, showing that this is locally free and hence flat over such ideal. On the other hand, if $b(t_0) = 0$ then α is not integral over $K[t]_\mathfrak{m}$. In general (as in the case of the cusp) we cannot do anything about this, but for