Grading geometrically.

The goal of this note is to explain that gradings correspond to $\mathbb{G}_m$-actions and that the Proj construction is exactly mimicking what we are used to in real and complex geometry when construction projective spaces.
This was given as an exercise to hand in in a previous years so proofs are credited to past students who wrote them.

1. Gradings and actions

In what follows we write

$$\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}]).$$

Note that $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ defined by $t \mapsto xy$ gives a map of schemes[1]

$$m\colon \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$$

which we call the *multiplication.*
A $\mathbb{G}_m$-action on an affine scheme $X = \mathrm{Spec}(A)$ is a map[2]

$$\mu\colon \mathbb{G}_m \times X \to X$$

such that the following diagram commutes[3]

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m \times X & \xrightarrow{\mathrm{id}\times\mu} & \mathbb{G}_m \times X \\ {\scriptstyle m\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\mu} \\ \mathbb{G}_m \times X & \xrightarrow{\quad\mu\quad} & X \end{array}$$

and if c denotes the map $c\colon X \to \mathbb{G}_m \times X$ induced by the evaluation a 1, then $\mu c = \mathrm{id}$.

1.1. **The correspondence.** The set of $\mathbb{Z}$-gradings on a ring A (meaning gradings that turn A into a graded ring) is in one to one correspondence with the set of $\mathbb{G}_m$-actions on $\mathrm{Spec}(A)$.
(Daniil) Let's use the equivalence of categories of rings and affine schemes. So to each diagram from the definition of the action of $\mathbb{G}_m$ on an affine scheme X corresponds one to one with a map of rings $\mu^*\colon A \to A[t, t^{-1}]$ such that

$$\begin{array}{ccc} A[x, x^{-1}, y, y^{-1}] & \xleftarrow[\mathrm{id}\otimes\mu^*]{} & A[t, t^{-1}] \\ {\scriptstyle (t\to xy)\otimes\mathrm{id}}\uparrow & & \uparrow{\scriptstyle\mu^*} \\ A[t, t^{-1}] & \xleftarrow[\mu^*]{\qquad} & A \end{array}$$

[1]We have $\mathbb{G}_m \times \mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[x, x^{-1}, y, y^{-1}]$.
[2]We have $\mathbb{G}_m \times X = \mathrm{Spec}(A[t, t^{-1}]$.
[3]We have $\mathbb{G}_m \times \mathbb{G}_m \times X = \mathrm{Spec}(A[x, y, x^{-1}, y^{-1}]$.

and the composition of μ with evaluation at 1 (denoted by c) is id.
Write $\mu(a) = \sum_i a_i t^i$. The commutativity of the diagram means

$$\sum_i a_i(yx)^i = ((t \to xy) \otimes \mathrm{id})\,(\sum_i a_i t^i) = ((t \to xy) \otimes \mathrm{id})\,(\mu^*(a))$$

$$\overset{\text{commutativity of the diagram}}{=} (\mathrm{id} \otimes \mu^*)\,(\mu^*(a)) = (\mathrm{id} \otimes \mu^*)\,(\sum_{i=1}^{n} a_i t^i) = \sum_i \mu^*(a_i) x^i.$$

In other words, the diagram above commutes if and only if $\mu^*(a_i) = a_i y^i$ *i.e.* if and only if a_i belongs to $A_i = (\mu^*)^{-1}(At^i)$. As for the second condition, it holds if and only if $a = \sum_i a_i$.
It now follows that $A = \bigoplus A_i$. The sum is direct because μ is an injective map of abelian groups.
Note also that $A_i A_j \subset A_{i+j}$ because μ^* is a morphism of rings. We then see that this gives an associated grading on the ring A.
Now given a grading on $A = \oplus A_i$ where we denote $a = \sum a_i$ we see that $\mu^* \colon A \to A[t, t^{-1}]$ sending $a \mapsto \sum a_i t^i$ is a ring morphism and using translations in algebra of the two conditions of a $\mathbb{G}_m$-action, we see that μ^* is indeed one. These constructions are by construction inverse to each other.

1.2. **Homegeneous maps.** Let $d > 1$, S and R two $\mathbb{Z}$-graded rings, with associated action μ_S and μ_R. We show below that (where $(-)^d \colon \mathbb{G}_m \to \mathbb{G}_m$ is induced by Spec by the ring map determined by $t \mapsto t^d$)

$$\begin{array}{ccc} \mathbb{G}_m \times \mathrm{Spec}(S) & \xrightarrow{\mu_S} & \mathrm{Spec}(S) \\ {\scriptstyle ((-)^d, f)} \downarrow & & \downarrow {\scriptstyle f} \\ \mathbb{G}_m \times \mathrm{Spec}(R) & \xrightarrow{\mu_R} & \mathrm{Spec}(R) \end{array}$$

commutes[4] if and only φ factors through $S^{(d)} = \bigoplus_n S_{nd}$, meaning that is it *homogeneous of degree d*.
Translating this diagram to algebra gives the diagram

$$\begin{array}{ccc} S[t, t^{-1}] & \longleftarrow & S \\ {\scriptstyle t \mapsto t^d} \uparrow & & \uparrow {\scriptstyle \varphi} \\ R[t, t^{-1}] & \longleftarrow & R \end{array}$$

where the left vertical arrow is given by φ on the coefficients, whence that commuting correspond to φ sending elements of degree m to elements of degree md.

1.3. **Quotients.** Let A be a $\mathbb{Z}$-graded ring and $A_0 \to A$ the inclusion. Show that $\pi \colon \mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$ has the following universal property. For every

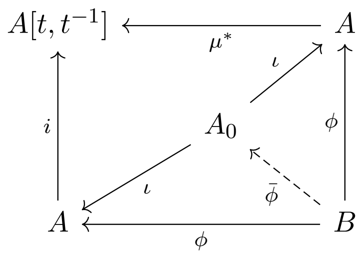
[4] On points, this means that $f(\lambda x) = \lambda^d f(x)$.

affine scheme X and map $f\colon \operatorname{Spec}(A) \to X$ with the property that

$$\begin{array}{ccc} \mathbb{G}_m \times \operatorname{Spec}(A) & \xrightarrow{\mu} & \operatorname{Spec}(A) \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(A) & \xrightarrow{f} & X \end{array} \qquad (\mathbb{G}_m\text{-invariant maps})$$

then there exists a unique map $\overline{f}\colon \operatorname{Spec}(A_0) \to X$ with $\overline{f}\pi = f$. It means by definition that $\pi : \operatorname{Spec}(A) \to \operatorname{Spec}(A_0)$ is the quotient by the action of $\mathbb{G}_m$ in the category of affine schemes.

(3)(Karl) Since X is an affine scheme, let's just say $X = \operatorname{Spec}(B)$. The morphism f correspond to some morphism $\phi : B \to A$. Let ι denote the inclusion $A_0 \to A$. The pr_2 is induced by the inclusion of A into $A[t, t^{-1}]$, which we'll call i. We want to show that there is a unique $\bar{\phi}$ such that the diagram below commutes.



We note that for $b \in B$, $i(\phi(b)) = \mu^*(\phi(b))$ if and only if $\phi(b) \in A_0$. So we can define $\bar{\phi} = \phi^{|A_0}$ which is necessarily unique because ι is injective.

2. The Proj construction as a quotient

Let B be a $\mathbb{N}$-graded ring. Using the above, let's explain the Proj construction as a quotient by $\mathbb{G}_m$.

In real geometry, the real projetive space of dimension n is defined as the quotient

$$\frac{\mathbb{R}^{n+1} \setminus 0}{\mathbb{R}^\times}$$

In what follows we explain why the Proj construction is exactly mimicking this in algebraic geometry. Namely, we have seen that a grading on an affine scheme $\operatorname{Spec}(B)$ uniquely corresponds to an action of $\mathbb{G}_m$. Note that $\mathbb{G}_m$ is the algebrico geometric incarnation of the multiplicative group $R^\times$.

When the grading is $\mathbb{N}$-graded, the ideal $B_+ = \sum_{i \geq 1} B_i$ is analogous to zero in $\mathbb{R}^{n+1}$, meaning that $V(B^+)$ is the analogue of zero.[5] So we are looking at the scheme

$$U = \operatorname{Spec}(B) \setminus V(B_+) = \bigcup_{\substack{b \in B^+ \\ \text{homogeneous}}} D(b).$$

This scheme is covered by the affine schemes $D(b) = \operatorname{Spec}(B_b)$. Because each b is homegeneous, each B_b is a $\mathbb{Z}$-graded ring. Because we covered U

[5]To be more precise, what can run an analogous analysis about action by the multiplcative monoid $\mathbb{A}^1$ and $\mathbb{N}$-gradings. When doing this one can formalize that $0 \cdot \operatorname{Spec}(B) = V(B^+)$, where $\cdot$ denotes the action.

by invariant opens by the $\mathbb{G}_m$-action, taking the quotient amounts to taking the quotient on each invariant open. Therefore the quotient

$$\frac{\mathrm{Spec}(B) \setminus V(B_+)}{\mathbb{G}_m} = \bigcup_{\substack{b \in B^+ \\ \text{homogeneous}}} \mathrm{Spec}(B_{(b)})$$

where the right hand side is to be understood as a gluing of affine schemes, which corresponds to the gluing in the Proj construction, so we sum up the discussion as

$$\frac{\mathrm{Spec}(B) \setminus V(B_+)}{\mathbb{G}_m} = \mathrm{Proj}(B).$$